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Malati Kesaree Charyulu
Iowa State University

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THEORETICAL STRESS DISTRIBUTION
IN AN ELASTIC MULTI-LAYERED MEDIUM

by

Malati Kesaree Charyulu

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Signature was redacted for privacy.

Head of Major Department

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Dean of Graduate College

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INTRODUCTION

Object and Scope

The unprecedented expansion of major expressway systems makes imperative the development and use of adequate methods for evaluation, design and construction control of flexible pavement systems. The problem to be solved by the engineer in highway design or airport construction deals primarily with layered soil deposits. In foundation engineering, serious problems are encountered, where a soft compressible clay layer is sandwiched at some depth between an upper layer of sand and underlying layer of sand or rock. Whether the surface layer is stronger or weaker than the underlying layer, sound foundation design of most important structures in civil engineering deals with the evaluation of stresses and displacements in multi-layered systems.

This investigation was undertaken to provide a general analysis of stresses and displacements of a multi-layered system for the engineer. The analysis is intended as a useful tool which can be directly applied to the analysis of actual conditions encountered in layered soil structures. Numerical evaluation of these quantities for certain cases is given in the form of influence curves for the solution of practical problems.

Review of Literature

An analysis of the stresses and displacements in a homogeneous, isotropic, elastic, semi-infinite media subjected to a vertical point load was first given by Boussinesq in 1885 (1). The stress function developed strictly satisfies the boundary conditions. The Boussinesq equations are given in a general form so that any value of Poisson's ratios can be substituted in them. But it is interesting to note that the vertical stress is independent of Poisson's ratio. In Boussinesq's work and in all the work done up to the present time the unit weight of the elastic material is assumed to be zero, so that the computations furnish only the stresses due to the surface loads. Therefore, to obtain the total stresses in an elastic material with a unit weight γ, produced by the surface loads and the weight of the elastic material, it is necessary to add the stresses due to the loads to those produced by the weight of the material. These stresses are

$$\sigma_z' = z\gamma$$

$$\sigma_r' = \sigma_\theta' = K_o\ z\gamma$$

$$\tau_{rz} = 0$$

where σ_z', σ_r', σ_θ' are the vertical, radial and circumferential stresses at depth z due to the weight of the material and

K_o is the coefficient of earth pressure at rest for the lateral earth pressure in the semi-infinite solid.

As quoted by Love (2), Cerruti derived equations for the stresses due to a horizontal point load acting on a horizontal surface which are not as simple as Boussinesq's equations. The computation of stresses due to a vertical and a horizontal force acting at a point beneath a horizontal surface derived by Mindlin (3) in 1936 are still more cumbersome. In order to use these equations they must be simplified at the expense of accuracy. Quoting from Terzaghi (4, p. 375):

> "With increasing depth below the surface the state of stress represented by Mindlin's equations approaches that which is produced by a force acting at a point in the interior of an infinite solid. The corresponding stress equations have been derived by Kelvin (about 1850). Introducing the special value $\mu = 0.5$ (Poisson's ratio for incompressible elastic solids) into his equations one finds that the stresses produced by the point load Q applied at a given point within an infinite solid are equal to one half of the stresses acting at the same point in a semi-infinite solid whose plane surface passes through the point of application of Q at a right angle to the direction of Q. Hence one obtains these stresses by dividing the stresses determined by Boussinesq's equations by two provided $\mu = 0.5$."

The equations for stresses in a semi-infinite media under infinite line load is obtained by integration of Boussinesq's stress equations. Since this is a problem of plain strain these stresses are independent of Poisson's ratio. Since in a perfectly elastic medium superposition of stresses due to different loads is valid, an infinite strip load can be considered as consisting of an infinite number of discreet point loads qdA. Using Boussinesq's equations as a starting point and integrating over the loaded area, stresses under infinite strip loads can also be obtained. Newmark obtained influence charts for stresses (5) and displacements (6) under surcharges covering rectangular areas. Foster and Ahlvin presented charts (7) for computing the influences due to circular loads. Different surcharges resembling railroad embankments, which are infinite strips with inclined slopes have also been analyzed. Gray (8) has compiled the equations for normal stresses under the weight of these surcharges on horizontal sections. He also gives a valuable list of references to the original publications. Tables and charts for these stresses have been prepared by Jërgenson (9).

All the above theories were developed with an assumption of perfect homogeneity and isotropy of the semi-infinite solid with respect to its elastic properties, although this condition seldom exists in nature. The most common deviations from the ideal state of elastic isotropy and homogeneity are due to the stratifications or laminations which are characteristic of all sedimentary deposits or due to a rapid decrease of compressibility with depth, which is typical of sandy soils. The average coefficient of permeability when measured parallel to the planes of stratification is greater than the coefficient of permeability normal to these planes for stratified soils. The ideal substitute for such a mass of soil with thin bedding planes is a semi-infinite, homogeneous but orthotropic elastic solid whose modulus of elasticity has a constant value (E_h) in every horizontal direction and a smaller value (E_v) in the vertical direction. Assuming that the ratio E_h/E_v is equal to an empirical constant "n" Wolf (10) computed the stresses produced by a point load and by a flexible strip load of infinite length. It can be seen that the stress decreases much more rapidly with depth for high values of "n" than it does for low values. The curves are identical with those of Boussinesq in which "n" equals one.

Westergaard (11) investigated the influence of laminations on the distribution of stresses in a different way. He assumed that the semi-infinite solid is reinforced by perfectly flexible horizontal membranes, which completely prevent any deformation in a horizontal direction without interfering with deformations in a vertical direction. If for the material located between the membranes Poisson's ratio is assumed to be zero, the vertical stress curve falls between the limiting cases of $n = 1$ and ∞ represented by the ideal cases of Boussinesq and of Wolf considering the modulus of elasticity in the horizontal direction as infinite. Another important deviation from Boussinesq's ideal elastic solid is a decrease in compressibility with increasing depth which is a development typical in cohesionless sands. For linearly elastic materials the vertical strain is independent of initial hydrostatic pressure, whereas for sands the strain due to the load decreases with increasing hydrostatic pressure. In a sand stratum the sand is under the influence of an hydrostatic pressure, due to the weight of the sand, the intensity of which increases with the depth below the surface. The strain produced by a given intensity of vertical stress in the sand decreases with increasing depth below the surface. In order to take this property of sands into account without losing the simplicity resulting from assuming the validity of the law of superpositions, it is assumed that the sand strictly obeys Hooke's law and that the modulus of elasticity of the sand increases with depth according to a definite law. In other words sand is assumed to be perfectly elastic and isotropic in every horizontal direction but elastically nonhomogeneous in a vertical direction. Griffith (12) proposed a semi-empirical modification in terms of a "concentration index" to Boussinesq's equation for taking care of the nonhomogeneous nature of sand.

In many locations, layers of naturally occurring soils are of finite depth and rest on comparatively rigid bases. The distribution of pressure in such masses can be computed using either of the following assumptions: (a) neither friction nor adhesion exists between the elastic layer and the base or (b) perfect adhesion exists between the layer and the base. The equations for the pressures produced by a point load on a layer with a frictionless base were developed by Melan (13) and those for adhesive base were developed by Biot (14). The computation of pressures due to line loads were found for the frictionless case by Melan (13) and for the adhesive case by Maguerre (15). Melan estimated that the greatest vertical stress on the rigid base occurs under the load, and this value for the frictionaless case is 71 percent higher than that of semi-infinite medium for point load. For adhesive assumptions it is 44 percent higher according to Marguerre. Stresses on the rigid base under the line loads are 56 and 28 percent higher than those at the same depth in homogeneous semi-infinite solids for frictionless and adhesive assumptions respectively.

The rigorous computation of the intensity and distribution of the vertical stress inside an elastic layer under a uniformly distributed load is more difficult. Cummings (16) solved this problem by obtaining the stress equation for points under the center line of the loaded area, and evaluating the stress at other points on the adhesive base by assuming that the shape of the stress curves for this case is similar to those that represent the distribution of normal stresses on horizontal sections through semi-infinite masses acted upon by the same loads.

In the nineteen forties Burmister (17, 18) formulated the general solution for a system having two or three layers for the case of axial symmetry, and gave numerical data for vertical displacements at the surface of a two-layer system subjected to a uniformly distributed load over a circular area. Corresponding data for stresses along the axis of symmetry have been published by Fox (19) and some results for a three-layer system by Acum and Fox (20). All the work on multi-layer systems has been accomplished by assuming perfectly elastic, homogeneous and isotropic layers and even these idealized conditions give rise to rather cumbersome equations. The assumption made for the interface is that there is either no friction between layers or that there is perfect adhesion between the layers.

Some experimental work in this field is now being done by the use of mathematical analogs and photoelastic models. But problems in three dimensions are difficult to solve by these methods due to the inherent two dimensional natures of the models. Field data is also being collected and compared with theoretical values but the validity of the theory cannot be truly evaluated since the field data depend on non-ideal materials. The use of pressure cells greatly alters the stresses and strains in the soil by virtue of volume displacement of the soil by the cell and the difference in properties of the soil and the cell.

The Present Investigation

This investigation deals with the analysis of a semi-infinite layered elastic medium subject to axially symmetrical static forces at the surface. The stresses due to the body forces are not considered since they can be added separately. The theory is first developed for an arbitrary number of horizontal layers. The material in any one layer is assumed to be homogeneous, isotropic and linearly elastic. The geometry of individual layers and the physical properties of the material may vary from layer to another and the lowest layer is considered to be semi-infinite.

The problem is solved for the realistic case of natural environmental conditions which dictate the existence of friction at the interface. The conditions assumed for the problem are different than those assumed in existing solutions and are based on the performance of pavements. Experience has shown that slippage occurs between elements of a pavement.

A flexible pavement consists of a wearing surface, base course, subbase and compacted subgrade. A layered pavement system is inherently a prestressed structure. It is constructed in sequence by preconditioning each layer.

The subgrade is first prepared by heavy rolling, a subbase layer is then placed and compacted on the subgrade. The subgrade layer is now effectively confined and restrained at the interface. Then a base course and a wearing surface is placed above them. It is well known that much of the time the wearing course does not bind well to the base because the base course is not primed or it is exceedingly dirty. In such cases slipping occurs causing defects called shoving (21). These problems are also due to the horizontal movement of the base course over the subbase. To avoid the effects due to poor binding, wire meshes are sometimes used at the interface. Perfect mechanical bonding of the subbase and subgrade is difficult to attain and even if obtained during construction the adhesion at the interface is bound to be reduced during the life of the pavement due to repeated loads, climatic conditions, moisture changes and other factors. Burmister in his latest work (22) states:

> "However it is believed that the principal reason for the poor performances of the 6-inch base generally is that a single 6-inch base cannot be compacted properly by usual construction methods on a relatively poor and hence yielding subgrade, due to excessive 'weaving' of the subgrade under the rolling action. Therefore, due to yielding, the essential 'keying and mechanical bonding and prestressing' of the base course material could not be attained." "The six strength values below average - 6(8,000-11,000-14,000) in sections M, R and O represents very poor subbase strength properties and are evidence of poor shear deformation characteristics

> due to inadequate mechanical bonding, keying, and prestressing of the subbase material during construction."

It is therefore more reasonable to assume that there is relative displacement in the horizontal direction and that friction exists at all interfaces than to assume either frictionless sliding of one layer over the other or complete welding and adhesion of layers as done by all previous investigators.

In the design of pavements, accurate estimation of design loads is just as important as accurate estimation of stresses under those loads. It is known that the distribution of pressure under a tire, although generally considered uniform, is greatest on the center line of the tire imprint and becomes zero at the edges which can be seen by the intensity of imprint at the center and edges. Very little work has been done in this field on truck tires, but Lawton's graphs (23) of contact pressure versus distance from center of load indicate that the contact pressure distribution is approximately a parabola at or below the rated tire load. When overloaded the tire tread tends to buckle, and the pressure pattern approaches one of uniform distribution.

The analysis of this investigation therefore assumes a multi-layered system, consisting of homogeneous and isotropic layers with physical properties that differ from layer to layer. The system is subjected to parabolic distribution of pressure on a circular surface area. The interface conditions considered are continuity of vertical stress and displacements and continuity of shear stress across the interface. The shear stress is taken to be proportional to the relative displacement at the interface, which is true for silty and sandy soils. This relation between shear stress and relative displacement at the interface has been experimentally verified by Terzaghi and Peck (24).

Numerical values of vertical stresses and displacements, for direct use of the engineer, have been computed by a high speed electronic computer, "I.B.M. 7074", for a four-layered medium, both for a gravel subgrade and a sandy subgrade. The elastic constants for the pavement materials and the subgrades have been taken from Nijboer (25).

The present computation program is set up in such a way that the stresses and displacements can be evaluated for any combination of elastic constants and physical properties of the pavement layer, different proportionality constants between the shear stress and relative displacement at different interfaces, and any parabolic shape of the distribution of contact pressure on the surface of the top layer. The extent to which these computed effects approximate actual effects depends entirely on how closely the conditions of actuality conform to those assumed in this analysis. The most unreliable portion of the analysis lies in the dependability of the interfacial shear-displacement data. It is hoped that improved estimates of the relationship can be experimentally determined. Stresses and displacements can then be evaluated for any situation in question by use of the program used for this analysis.

ANALYSIS OF THE PROBLEM

Statement of the Problem

Highways are constructed in a series of layers of different materials in order to distribute high surface stresses to relatively low bearing subgrades. Generally the highest quality materials are located near the surface. The load carrying capacity of such a highway is dependent upon the load-distributing characteristics of the layered system. In order to create a rational economic design of a layered pavement system it is necessary to understand the load-distributing characteristics of the system. Most pavements consist of a surface course, a base course, a subbase course and a subgrade. The thickness of layers, types of materials and other variables offer an infinite number of possible combinations. Several simplifying assumptions must be made in order to arrive at a theoretical mathematical description of the stresses induced in the components of a highway by tire loads on the surface. Under such idealized assumptions a highway pavement can be represented by a layered system as described below.

Consider the layered system shown in Figure 1. It consists of a number of horizontal layers of uniform thickness and infinite extent in the horizontal plane. Each

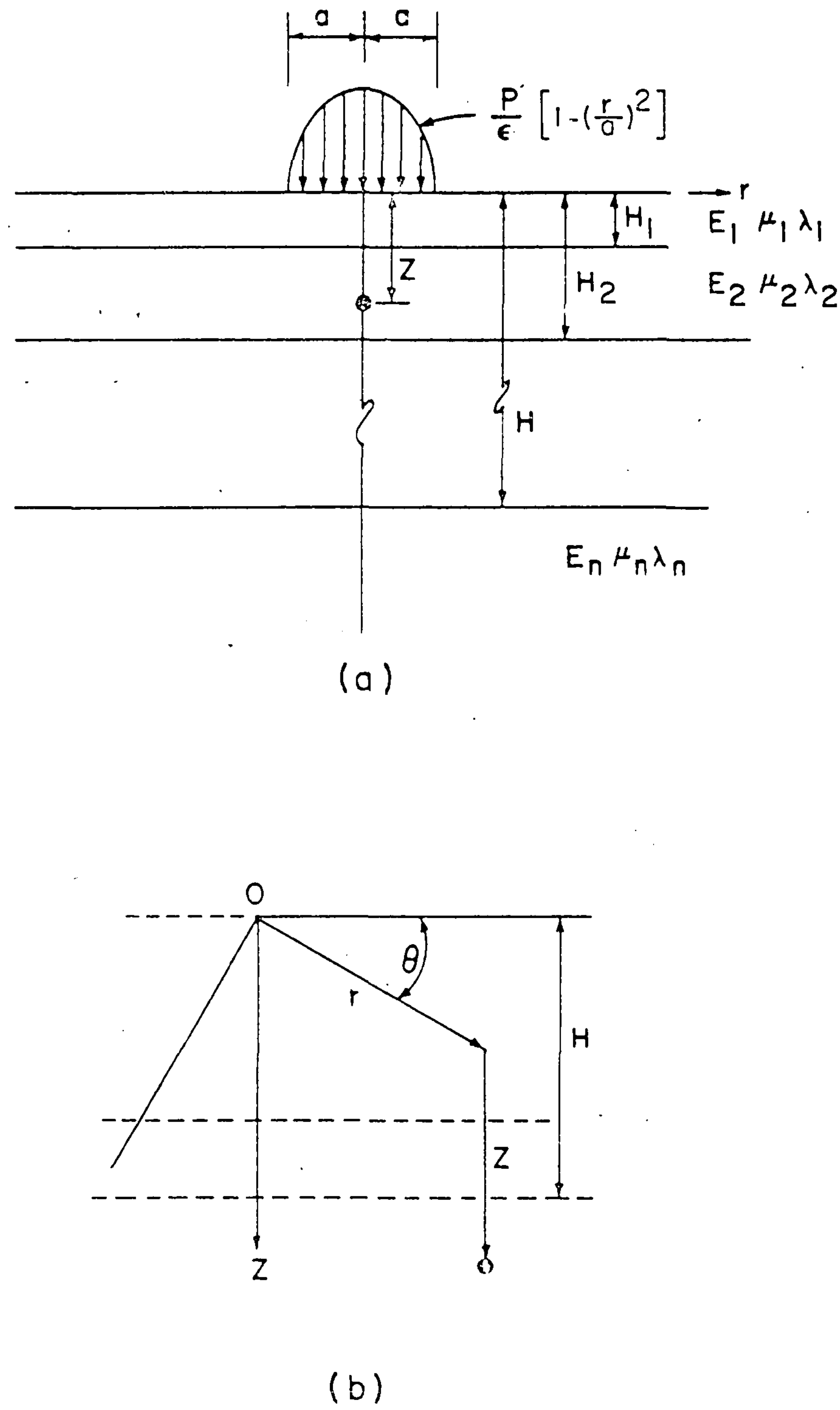

Figure 1. The layered system

layer is assumed to be homogeneous, isotropic and linearly elastic. The thickness of the several layers and their elastic properties are different. The lower layer is assumed to extend to infinity in the horizontal and vertical directions. The surface forces representing the tire contact pressure are considered to be strictly axially symmetric. The forces caused by gravity have not been considered.

Cylindrical co-ordinates (r, θ, z) are used because of the axially symmetric nature of the system in question. The origin is located at the surface of the upper layer and the z axis coincides with the line of symmetry and is positive downward. The solution of the problem is based on the linear theory of elasticity.

The general method of analysis essentially involves the determination of a stress function for each layer. The stresses and displacements for each layer are expressed in terms of its stress function, which satisfies the boundary conditions of the particular layer. The layered system is subjected to boundary conditions at the upper layer and infinity as well as at all interfaces.

It has been found to be most convenient to substitute the compatibility equations, written only in terms of the strain components, into the equilibrium equations and determine a function which satisfies the resulting differential equation and the assumed boundary conditions. The stresses and displacements can then be written in terms of this function.

Development of the Problem

Stress-strain relations

In order to develop the problem in cylindrical coordinates, we refer to an element with displacements as indicated in Figure 2. Due to axial symmetry, no displacements in the θ direction are assumed. There are three normal strains ϵ_r, ϵ_θ and ϵ_z and three shearing strains $\gamma_{r\theta}$, $\gamma_{\theta z}$ and γ_{zr}. Considering first the displacement u in the r direction, from Figure 2(a) we obtain

$$\epsilon_r = \frac{\left[u + (\frac{\partial u}{\partial r})\ dr - u\right]}{dr} = \frac{\partial u}{\partial r}$$

It can also be seen that a pure radial displacement causes a strain in the θ direction, since the fibers of the element have elongated in the θ direction. The length of fiber ab was originally $rd\theta$; but, after the radial displacement u has taken place, the fiber is in the position of a'b' and the new length is $(r+u)d\theta$. The tangential strain due to this radial displacement is, therefore,

$$\epsilon_\theta = \frac{(r+u)\ d\theta - rd\theta}{rd\theta} = \frac{u}{r}$$

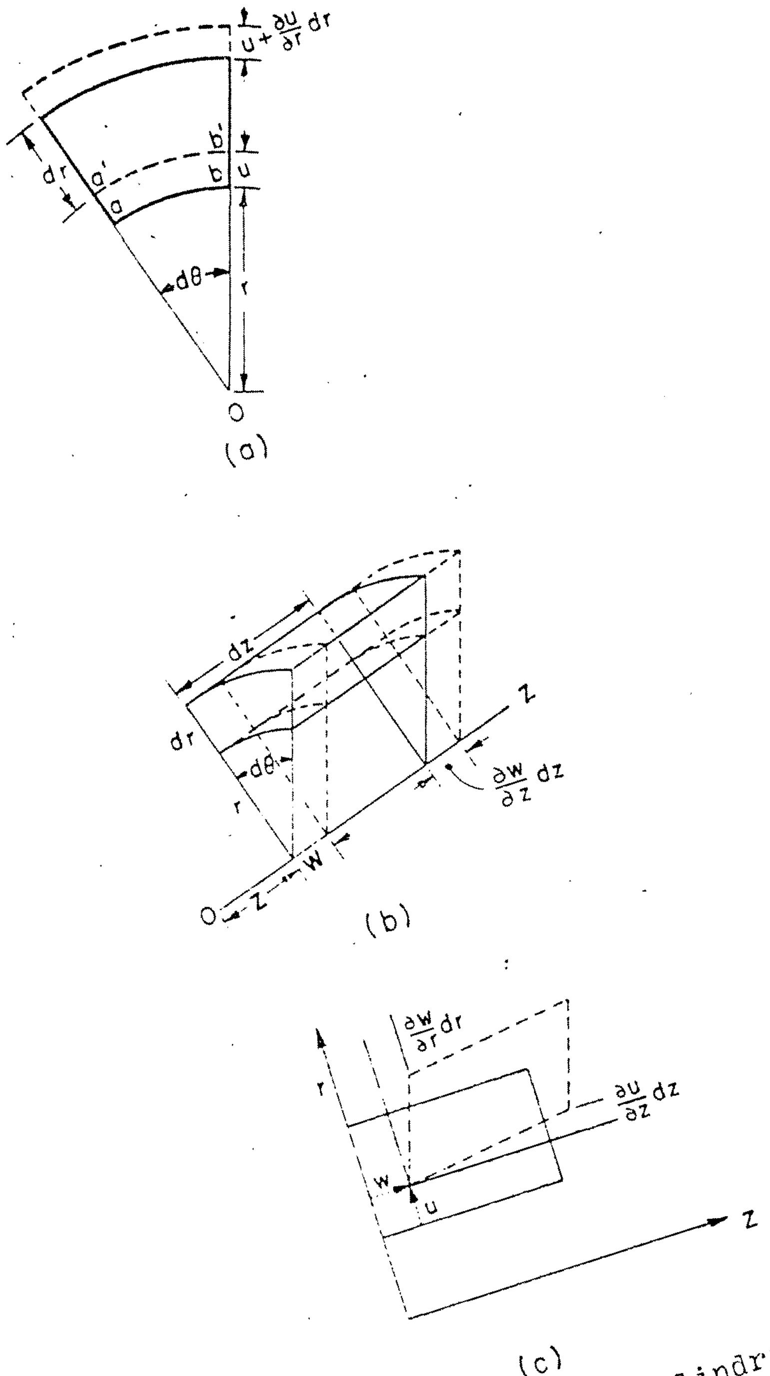

Figure 2. Displacements in cylindrical co-ordinates

The normal strain in the axial or z direction is given by

$$\epsilon_z = \frac{\partial w}{\partial z}$$

as in the case of the rectangular co-ordinate system. Since it is assumed that there are no tangential displacements and that all quantities are independent of θ the two shearing strains $\gamma_{\theta r}$ and $\gamma_{\theta z}$ are zero. From Figure 2(c) we obtain γ_{rz},

$$\gamma rz = \frac{(\partial u/\partial z)dz}{dz(1+\partial u/\partial z} + \frac{(\partial u/\partial r)\partial r}{(1+\partial u/\partial r)dr}$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

Therefore, the six components of strain in cylindrical co-ordinates with (u, o, w) displacements representing axial symmetry are

$$\epsilon_r = \frac{\partial u}{\partial r} \qquad (1)$$

$$\epsilon_\theta = \frac{u}{r}$$

$$\epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma\theta z = 0$$

$$\gamma_{\theta r} = 0$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

From Hooke's law, limiting the discussion to isotropic bodies, the stress-strain relations can be written as

$$\begin{aligned}
\epsilon_r &= \frac{1}{E}\left[\sigma_r - \nu\ (\sigma_\theta+\sigma_z)\right] \\
\epsilon_\theta &= \frac{1}{E}\left[\sigma_\theta - \nu\ (\sigma_r+\sigma_z)\right] \\
\epsilon_z &= \frac{1}{E}\left[\sigma_z - \nu\ (\sigma_\theta+\sigma_r)\right]
\end{aligned} \tag{3}$$

$$\begin{aligned}
\gamma_{\theta z} &= 0 \\
\gamma_{\theta r} &= 0 \\
\gamma_{rz} &= \frac{2(1+\nu)}{E}\ \tau_{rz}
\end{aligned} \tag{4}$$

where E and ν are the modulus of elasticity and Poisson's ratio respectively.

By algebraic manipulation, Equations 3 and 4 can also be written as

$$\begin{aligned}
\sigma_r &= \lambda\ (\epsilon_r + \epsilon_\theta + \epsilon_z) + 2\mu\epsilon_r \\
\sigma_\theta &= \lambda\ (\epsilon_r + \epsilon_\theta + \epsilon_z) + \sigma\mu\epsilon_\theta \\
\sigma_z &= \lambda\ (\epsilon_r + \epsilon_\theta + \epsilon_z) + 2\mu\epsilon_z
\end{aligned} \tag{5}$$

$$\begin{aligned}
\tau_{\theta z} &= 0 \\
\tau_{\theta r} &= 0 \\
\tau_{rz} &= \mu\ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)
\end{aligned} \tag{6}$$

where λ and μ are Lame's constants, and their values are

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad \mu = \frac{E}{2\ (1+\nu)}$$

Equilibrium equations

The distribution of stresses in a solid of revolution deformed symmetrically with respect to the axis of revolution is shown in Figure 3. The state of stress at any point of the solid is uniquely specified by four components of stress σ_r, σ_θ, σ_z and τ_{rz}. If we consider the total force in the radial direction, we must take into account the fact that the stress components σ_θ on each face of the element give rise to a force $-\sigma_\theta d\theta dr\ dz$ in the r direction as shown in Figure 3b. We therefore obtain the equilibrium condition

$$\begin{aligned} &(\sigma_r + \tfrac{1}{2}\frac{\partial\sigma_r}{\partial r}\,dr)(r + \tfrac{1}{2}\,dr)\ d\theta\ dz \\ &- (\sigma_r - \tfrac{1}{2}\frac{\partial\sigma_r}{\partial r}\,dr)(r - \tfrac{1}{2}\,dr)\ d\theta dz \\ &- \sigma_\theta\ d\theta\ dz \cdot dr + (\tau_{rz} + \tfrac{1}{2}\frac{\partial\tau_{rz}}{\partial z}\,dz) r dr\ d\theta \\ &- (\tau_{rz} - \tfrac{1}{2}\frac{\partial\tau_{rz}}{\partial z}\,dz) r\ dr\ d\theta = 0 \end{aligned}$$

Assuming that there are no body forces present and that the element, which is centered at the point (r, θ, z) has the dimensions (dr, r$d\theta$, dz), in the limit as dr, $d\theta$ and dz

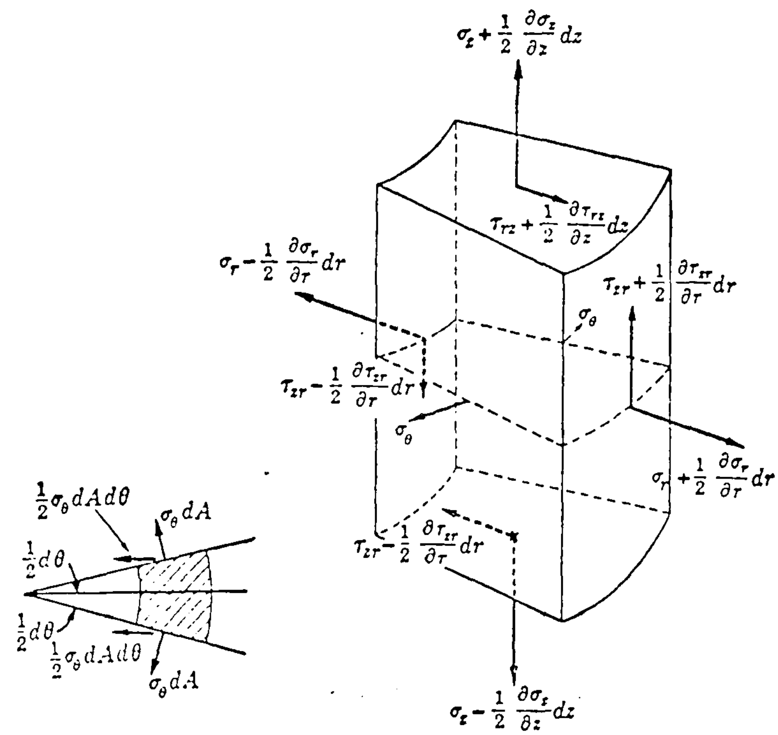

Figure 3. Stresses acting on an element of a solid of revolution

approach zero, the equation of equilibrium becomes

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (7)$$

Similarly, if we equate the total force in the z direction to zero, the second equation of equilibrium is obtained.

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \qquad (8)$$

Compatibility equations

Since the derivation of the compatibility equations in cylindrical co-ordinates is very cumbersome, the equations will be derived in cartesian co-ordinates and then transformed into cylindrical co-ordinates.

In rectangular co-ordinates the stress-strain relations for a displacement vector (u_x, v_y, w_z) are given as

$$\begin{aligned} \epsilon_x &= \frac{\partial u_x}{\partial x} \\ \epsilon_y &= \frac{\partial v_y}{\partial y} \\ \epsilon_z &= \frac{\partial w_z}{\partial z} \end{aligned} \qquad (9)$$

$$\begin{aligned} \gamma xy &= \frac{\partial u_x}{\partial y} + \frac{\partial v_y}{\partial x} \\ \gamma yz &= \frac{\partial v_y}{\partial z} + \frac{\partial w_z}{\partial y} \\ \gamma zx &= \frac{\partial u_x}{\partial z} + \frac{\partial w_z}{\partial x} \end{aligned} \qquad (10)$$

where ϵ_x, ϵ_y, ϵ_z are the normal strains and γ_{xy}, γ_{yz}, γ_{zx} are the shear strains.

Compatibility conditions are relations obtained by eliminating displacements from relations between displacements and strains. They express the conditions necessary and sufficient for the determination of the displacements from strains except for rigid body motion. The following compatibility equations are obtained from Equations 9 and 10

$$\frac{\partial^2 \epsilon x}{\partial y^2} + \frac{\partial^2 \epsilon y}{\partial x^2} = \frac{\partial^2 \gamma xy}{\partial x \partial y}$$

$$\frac{\partial^2 \epsilon y}{\partial z^2} + \frac{\partial^2 \epsilon z}{\partial y^2} = \frac{\partial^2 \gamma yz}{\partial y \partial z} \qquad (11)$$

$$\frac{\partial^2 \epsilon z}{\partial x^2} + \frac{\partial^2 \epsilon x}{\partial z^2} = \frac{\partial^2 \gamma zx}{\partial z \partial x}$$

$$2 \frac{\partial^2 \epsilon x}{\partial y \partial z} = \frac{\partial}{\partial x} \left(-\frac{\partial \gamma yz}{\partial x} + \frac{\partial \gamma xz}{\partial y} + \frac{\partial \gamma xy}{\partial z}\right)$$

$$2 \frac{\partial^2 \epsilon y}{\partial x \partial z} = \frac{\partial}{\partial y} \left(\frac{\partial \gamma yz}{\partial x} - \frac{\partial \gamma xz}{\partial y} + \frac{\partial \gamma xy}{\partial z}\right) \qquad (12)$$

$$2 \frac{\partial^2 \epsilon z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial \gamma yz}{\partial x} + \frac{\partial \gamma xz}{\partial y} - \frac{\partial \gamma xy}{\partial z}\right)$$

To transform these compatibility equations (expressed in terms of strains) to equations expressed in terms of stresses we substitute the stress-strain relations into the above equations. The stress-strain equations in cartesian co-ordinates are

$$\epsilon_x = \frac{1}{E}\left[(1 + \nu)\, \sigma_x - \nu\Phi\right]$$
$$\epsilon_y = \frac{1}{E}\left[(1 + \nu)\, \sigma_y - \nu\Phi\right] \qquad (13)$$
$$\epsilon_z = \frac{1}{E}\left[(1 + \nu)\, \sigma_z - \nu\Phi\right]$$

$$\gamma xy = 2\, \frac{(1 + \nu)}{E}\, \tau xy$$
$$\gamma yz = 2\, \frac{(1 + \nu)}{E}\, \tau yz \qquad (14)$$
$$\gamma zx = 2\, \frac{(1 + \nu)}{E}\, \tau zx$$

where $\Phi = \sigma_x + \sigma_y + \sigma_z$.

Substituting ϵ_x, ϵ_y and γ_{xy} in terms of stresses in the first compatibility equation we obtain

$$(1+\nu)\left(\frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial x^2}\right) - \nu\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) = 2(1+\nu)\, \frac{\partial^2\tau_{xy}}{\partial x\partial y} \qquad (15)$$

Equation 15 can be simplified further by making use of equilibrium equations which, in cartesian co-ordinates, when no body forces are considered, can be written as

$$\frac{\partial \tau yz}{\partial y} = - \frac{\partial \sigma_z}{\partial z} - \frac{\partial \tau xz}{\partial x}$$

$$\frac{\partial \tau xy}{\partial x} = - \frac{\partial \sigma_y}{\partial y} - \frac{\partial \tau yz}{\partial z} \qquad (16)$$

$$\frac{\partial \tau xy}{\partial y} = - \frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau xz}{\partial z}$$

Differentiating the second equation with respect to y and the third with respect to z and adding we get

$$2 \frac{\partial^2 \tau xy}{\partial x \partial y} = - \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial}{\partial z} \left(\frac{\partial \tau yz}{\partial y} + \frac{\partial \tau xz}{\partial x}\right) \qquad (17)$$

Substituting for $\frac{\partial \tau yz}{\partial y}$ again from Equation 16, Equation 17 becomes

$$2 \frac{\partial^2 \tau xy}{\partial x \partial y} = - \left[\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z^2}\right] \qquad (18)$$

Equation 15 can now be simplified to

$$(1 + \nu)(\nabla^2 \Phi - \nabla^2 \sigma_z - \frac{\partial^2 \Phi}{\partial z^2}) - \nu \, (\nabla^2 \Phi - \frac{\partial^2 \Phi}{\partial z^2}) = 0$$

or $\qquad (19)$

$$(\nabla^2 \Phi - \nabla^2 \sigma_z - \frac{\partial^2 \Phi}{\partial z^2}) - \nu \, (\nabla^2 \sigma_z) = 0$$

where the operator ∇^2 is used to represent $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$.

Equations similar to 19 can be written for each of the three Equations 11 which when added become

$$(1 - \nu)\nabla^2\Phi = 0$$

or

$$\nabla^2\Phi = 0$$

Hence Equation 19 can be further simplified to

$$(1 + \nu)\nabla^2\sigma_z + \frac{\partial^2\Phi}{\partial z^2} = 0 \qquad (20)$$

Using similar methods, the three strain equations of compatibility (11) can be transformed to the three stress equations of compatibility, as given below

$$(1 + \nu)\nabla^2\sigma_z + \frac{2^2\Phi}{\partial z^2} = 0$$

$$(1 + \nu)\nabla^2\sigma_x + \frac{\partial^2\Phi}{\partial x^2} = 0 \qquad (21)$$

$$(1 + \nu)\nabla^2\sigma_y + \frac{\partial^2\Phi}{\partial y^2} = 0$$

To transform Equations 12 substitute values for γ_{xy}, γ_{yz}, γ_{zx} and ϵ_x from 13 and 14. This gives us

$$\left[(1+\nu)\frac{\partial^2\sigma x}{\partial y\partial z} - \nu\frac{\partial^2\Phi}{\partial y\partial z}\right] = (1+\nu)\left[-\frac{\partial^2\tau_{yz}}{\partial x^2} + \frac{\partial^2\tau_{xz}}{\partial x\partial y} + \frac{\partial^2\tau_{xy}}{\partial x\partial z}\right]$$

Values of $\frac{\partial^2 \tau_{xz}}{\partial x \partial y}$ and $\frac{\partial^2 \tau_{xy}}{\partial x \partial z}$ are obtained by differentiating the first equation of 16 with respect to y and the second with respect to z respectively. Substituting these values in the above equation we get

$$(1+\nu)\frac{\partial^2 \sigma_x}{\partial y \partial z} - \nu \frac{\partial^2 \Phi}{\partial y \partial z} = (1+\nu)\left[-\frac{\partial^2 \tau_{yz}}{\partial x^2} - \frac{\partial^2 \tau_{yz}}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z \partial y} - \frac{\partial^2 \tau_{yz}}{\partial z^2} - \frac{\partial^2 \sigma_y}{\partial y \partial z}\right]$$

$$(1+\nu)\frac{\partial^2 \Phi}{\partial y \partial z} - \nu \frac{\partial^2 \Phi}{\partial y \partial z} = (1+\nu)\left[-\nabla^2 \tau_{yz}\right]$$

$$(1+\nu)\nabla^2 \tau_{yz} + \frac{\partial^2 \Phi}{\partial y \partial z} = 0$$

starting from the other two compatibility equations of 12 we can similarly derive two more compatibility equations in terms of stress. So the three equations are

$$(1+\nu)\nabla^2 \tau_{yz} + \frac{\partial^2 \Phi}{\partial y \partial z} = 0 \tag{22a}$$

$$(1+\nu)\nabla^2 \tau_{xy} + \frac{\partial^2 \Phi}{\partial x \partial y} = 0 \tag{22b}$$

$$(1+\nu)\nabla^2 \tau_{xz} + \frac{\partial^2 \Phi}{\partial x \partial z} = 0 \tag{22c}$$

Equations 21 and 22 represent the six equations of compatibility, called Beltrami-Michell equations.

<u>Compatibility equations in cylindrical co-ordinates</u>

The relationships between normal and shear stresses in cartesian co-ordinates and cylindrical co-ordinates are first obtained because these are required for the transformation of compatibility equations. Since the z-axis is identical in both systems it is sufficient to use the two dimensional diagram given in Figure 4. Knowing the stress components σ_x, σ_y, τ_{xy} at any point '0', the stress acting in any direction r can be calculated by the equations of statics. If θ is the angle between r and x, taking a small element OCB whose depth in the z-direction is unity, the components of normal and shear stresses acting on OC and OB can be written as

$$\sigma_x \text{ BC } \cos\theta \qquad \text{and} \qquad \sigma_y \text{ BC } \sin\theta$$

$$\tau_{yx} \text{ BC } \cos\theta \qquad \text{and} \qquad \tau_{xy} \text{ BC } \sin\theta$$

respectively. Adding the forces, and noting that $\tau_{yx} = \tau_{xy}$ the equilibrium condition gives

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

and

$$\tau_{r\theta} = \tau_{xy} (\cos^2\theta - \sin^2\theta) + (\sigma_y - \sigma_x) \sin\theta \cos\theta$$

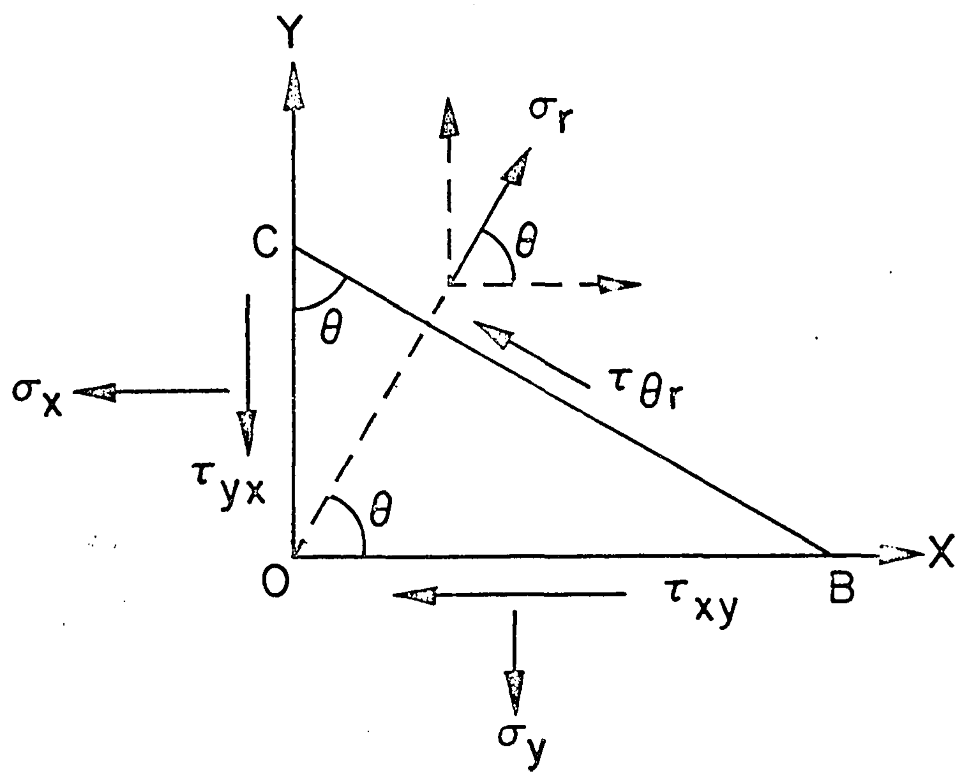

Figure 4. Stresses acting on an element

But for the problem under consideration $\tau_{r\theta} = 0$. Therefore τ_{xy} $(\cos^2\theta - \sin^2\theta) + (\sigma_y - \sigma_x) \sin\theta \cos\theta = 0$ (23)
which gives

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + (\sigma_x - \sigma_y) \frac{2 \sin^2\theta \cos^2\theta}{\cos^2\theta - \sin^2\theta} \tag{24}$$

Knowing σ_x, σ_y, τ_{xy} we can also find the stresses σ_θ and $\tau_{\theta r}$ by using the above method. Since the angle that σ_θ makes with the x axis is $(\theta+90^o)$ and $\tau_{\theta r}$ is zero, substitution of $(\theta+90^o)$ for θ in Equation 24 gives

$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - (\sigma_x - \sigma_y) \frac{2 \sin^2\theta \cos^2\theta}{\cos^2\theta - \sin^2\theta} \tag{25}$$

Multiplying both sides of Equations 24 and 25 by $(\cos^2\theta - \sin^2\theta)$ we obtain

$$\sigma_r (\cos^2\theta - \sin^2\theta) = \sigma_x \cos^2\theta - \sigma_y \sin^2\theta$$

$$\sigma_\theta (\cos^2\theta - \sin^2\theta) = - \sigma_x \sin^2\theta + \sigma_y \cos^2\theta$$

Solving for σ_x and σ_y from the above equations we have

$$\begin{aligned} \sigma_x &= \sigma_r \cos^2\theta + \sigma_\theta \sin^2\theta \\ \sigma_y &= \sigma_r \sin^2\theta + \sigma_\theta \cos^2\theta \end{aligned} \tag{26}$$

By substituting Equations 26 in 23 we obtain

$$\tau_{xy}\ (\cos^2\theta - \sin^2\theta) + \sin\theta\ \cos\theta \left[(\sigma_r-\sigma_\theta)(\sin^2\theta - \cos^2\theta)\right] = 0 \quad .$$

or

$$\tau_{xy} = (\sigma_r-\sigma_\theta)\ \sin\theta\ \cos\theta = \frac{1}{2}\ (\sigma_r-\sigma_\theta)\ \sin 2\ \theta \qquad (27)$$

The following equations are found useful in changing from rectangular to cylindrical co-ordinates:

$$r^2 = x^2 + y^2 \qquad \theta = \text{arc tan}\ \frac{y}{x} \qquad z = z$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \qquad (28)$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

The chain rule of different iation states that if

$$\Phi = f(r,\ \theta)$$

then

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \Phi}{\partial \theta}\frac{\partial \theta}{\partial x}$$

Therefore from Equations 28

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\sin\theta \qquad (29)$$

$$\frac{\partial^2\Phi}{\partial x^2} = (\cos\,\theta\,\frac{\partial}{\partial r} - \frac{\sin\,\theta}{r}\,\frac{\partial}{\partial\theta})(\cos\,\theta\,\frac{\partial\Phi}{\partial r} - \frac{\sin\,\theta}{r}\,\frac{\partial\Phi}{\partial\theta})$$

$$= \frac{\partial^2\Phi}{\partial r^2}\cos^2\theta - 2\,\frac{\partial^2\Phi}{\partial r\partial\theta}\,\frac{\sin\,\theta\,\cos\,\theta}{r} + \frac{\partial\Phi}{\partial r}\,\frac{\sin^2\theta}{r}$$

$$+ 2\,\frac{\partial\Phi}{\partial\theta}\,\frac{\sin\,\theta\,\cos\,\theta}{r^2} + \frac{\partial^2\Phi}{\partial\theta^2}\,\frac{\sin^2\theta}{r^2} \qquad (30)$$

But since σ_r, σ_θ and σ_z are independent of θ in a problem of axial symmetry $\frac{\partial\Phi}{\partial\theta} = 0$ which gives

$$\frac{\partial^2\Phi}{\partial x^2} = \frac{\partial^2\Phi}{\partial r^2}\cos^2\theta + \frac{\partial\Phi}{\partial r}\,\frac{\sin^2\theta}{r} \qquad (31)$$

It can be shown by the preceding process that

$$\frac{\partial^2\Phi}{\partial y^2} = (\sin\,\theta\,\frac{\partial}{\partial r} + \frac{\cos\,\theta}{r}\,\frac{\partial}{\partial\theta})(\sin\,\theta\,\frac{\partial\Phi}{\partial r} + \frac{\cos\,\theta}{r}\,\frac{\partial\Phi}{\partial\theta})$$

$$= \frac{\partial^2\Phi}{\partial r^2}\sin^2\theta + 2\,\frac{\sin\,\theta\,\cos\,\theta}{r}\,\frac{\partial^2\Phi}{\partial r\partial\theta} + \frac{\partial\Phi}{\partial r}\,\frac{\cos^2\theta}{r}$$

$$- 2\,\frac{\partial\Phi}{\partial\theta}\,\frac{\sin\,\theta\,\cos\,\theta}{r^2} + \frac{\partial^2\Phi}{\partial\theta^2}\,\frac{\cos^2\theta}{r^2}$$

Adding this equation, Equation 30 and $\frac{\partial^2\Phi}{\partial z^2}$ we get the important result

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} \qquad (32)$$

The transformation of the stress equations of compatibility from cartesian co-ordinates to cylindrical co-ordinates can now be made.

Using the operator of Equation 32 on Equation 26

$$\nabla^2\sigma_x = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)(\sigma_r \cos^2\theta + \sigma_\theta \sin^2\theta)$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)(\sigma_r \cos^2\theta + \sigma_\theta \sin^2\theta)$$

$$- \frac{2}{r^2} \sin\theta \cos\theta \; (\sigma_r - \sigma_\theta) \qquad (33)$$

If Φ is the sum of three normal components of stress at any point then

$$\Phi = \sigma_x + \sigma_y + \sigma_z$$

or from Equation 26

$$\Phi = \sigma_r + \sigma_\theta + \sigma_z$$

Substituting Equations 33 and 31 in 21 we obtain

$$\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\sigma_r - \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\nu}\frac{\partial^2\Phi}{\partial r^2}\right]\cos^2\theta$$

$$+ \left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right.\sigma_\theta + \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\nu}\frac{1}{r}\frac{\partial\Phi}{\partial r}\right]\sin^2\theta = 0$$

This equation is valid for all values of θ since we have axial symmetry. The above equation is satisfied for all values of θ if and only if

$$\left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial z^2}\right)\sigma_r - \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\nu}\frac{\partial^2 \Phi}{\partial r^2} = 0 \qquad (34)$$

and

$$\left(\frac{\partial^2}{\partial r^2} \quad \frac{1}{r}\frac{\partial}{\partial r} \quad \frac{\partial^2}{\partial z^2}\right)\sigma_r + \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\nu}\frac{1}{r}\frac{\partial \Phi}{\partial r} = 0 \qquad (35)$$

If we substitute for $\frac{\partial^2 \Phi}{\partial y^2}$ and $\nabla^2 \sigma_y$ in the third equation of 21 we again get the same equations as 34 and 35. The first equation of 21 remains unchanged from cartesian to cylindrical co-ordinates since the independent variable is z in both cases. To transform the compatibility Equations 22 we recall Equations 6, and because θ is the angle between r and the x axis we have $u \cos\theta = u_x$, $u \sin\theta = v_y$, $w = w_z$.

$$\frac{\partial u}{\partial z} = \left(\frac{\partial u_x}{\partial z}\right)\left(\frac{1}{\cos\theta}\right) \qquad\qquad \frac{\partial u}{\partial z} = \left(\frac{\partial v_y}{\partial z}\right)\left(\frac{1}{\sin\theta}\right)$$

The chain rule of differentiation gives

$$\frac{\partial w}{\partial r} = \left(\frac{\partial w_z}{\partial x}\right)\left(\frac{\partial x}{\partial r}\right) \qquad \text{and} \qquad \frac{\partial w}{\partial r} = \left(\frac{\partial w_z}{\partial y}\right)\left(\frac{\partial y}{\partial r}\right)$$

Using these equations with Equations 6 and 28, τ_{rz} can be written as

$$\tau_{rz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial w_z}{\partial x}\right)\frac{1}{\cos\theta}$$

$$\tau_{rz} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial w_z}{\partial y}\right)\frac{1}{\sin\theta}$$

Referring to Equations 10 and using the stress-strain relations in cartesian co-ordinates

$$\tau_{rz} = \tau_{xz} \left(\frac{1}{\cos\theta}\right) \qquad \text{or} \qquad \tau_{xz} = \tau_{rz} \cos\theta$$

$$\tau_{rz} = \tau_{yz} \left(\frac{1}{\sin\theta}\right) \qquad \text{or} \qquad \tau_{yz} = \tau_{rz} \sin\theta$$

Because $\frac{\partial\Phi}{\partial\theta}$ is identically equal to zero we can write Equation 29 as

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial r} \cos\theta$$

which gives

$$\frac{\partial^2\Phi}{\partial x\partial z} = \frac{\partial^2\Phi}{\partial r\partial z} \cos\theta \qquad (36)$$

Also

$$\nabla^2\tau_{xz} = \nabla^2(\tau_{rz} \cos\theta) = \cos\theta\, \nabla^2\tau_{rz} + \tau_{rz} \nabla^2 (\cos\theta)$$

$$= \cos\theta\, \nabla^2\tau_{rz} + \frac{\tau_{rz}}{r^2} \frac{\partial^2}{\partial\theta^2} (\cos\theta)$$

$$= \left(\nabla^2\tau_{rz} - \frac{\tau_{rz}}{r^2}\right) \cos\theta \qquad (37)$$

Using Equations 36 and 37 in Equation 22c we obtain

$$\nabla^2\tau_{rz} - \frac{1}{r^2} \tau_{rz} + \frac{1}{1+\nu} \frac{\partial^2\Phi}{\partial r\partial z} = 0 \qquad (38)$$

Transformation of Equation 22a to cylindrical co-ordinates also results in Equation 38. For the transformation of 22b we proceed as follows:

As shown in the preceding transformation

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r} \cos \theta$$

differentiating with respect to y

$$\frac{\partial^2 \Phi}{\partial y \partial x} = \frac{\partial}{\partial r} \left(\frac{\partial \Phi}{\partial r} \cos \theta\right) \frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta} \left(\frac{\partial \Phi}{\partial r} \cos \theta\right) \frac{\partial \theta}{\partial y}$$

By Equation 28 and since $\frac{\partial^2 \Phi}{\partial \theta \partial r} = 0$

$$\frac{\partial^2 \Phi}{\partial y \partial x} = \frac{\partial^2 \Phi}{\partial r^2} \cos \theta \sin \theta - \frac{\partial \Phi}{\partial r} \frac{\sin \theta \cos \theta}{r}$$

If this relation and Equation 27 are substituted in Equation 22b

$$(1+\nu)\nabla^2 \left[\frac{1}{2} (\sigma_r - \sigma_\theta) \sin 2\theta\right]$$

$$+ \frac{\sin 2\theta}{2} \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r}\right) \Phi = 0$$

wiich is not an independent equation. It can also be obtained by subtracting Equation 35 from 34, because

$$\nabla^2 \left[\frac{1}{2} (\sigma_r - \sigma_\theta) \sin 2\theta\right] = \frac{\sin 2\theta}{2} \nabla^2 (\sigma_r - \sigma_\theta)$$

$$- \frac{4}{r^2} (\sigma_r - \sigma_\theta) \frac{\sin 2\theta}{2}$$

The six compatibility equations in cartesian co-ordinates are therefore reduced to the following four compatibility equations in cylindrical co-ordinates:

$$
\begin{aligned}
&\nabla^2\sigma_r - \frac{2}{r^2}(\sigma_r-\sigma_\theta) + \frac{1}{1+\nu}\frac{\partial^2\Phi}{\partial r^2} = 0 \\
&\nabla^2\sigma_\theta + \frac{2}{r^2}(\sigma_r-\sigma_\theta) + \frac{1}{1+\nu}\frac{1}{r}\frac{\partial\Phi}{\partial r} = 0 \\
&\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\Phi}{\partial z^2} = 0 \\
&\nabla^2\tau_{rz} - \frac{1}{r^2}\tau_{rz} + \frac{1}{1+\nu}\frac{\partial^2\Phi}{\partial r\partial\theta} = 0
\end{aligned}
\tag{39}
$$

Derivation of general differential equations

The development of the differential equation follows the method used by Love (26). To express the state of stress in the body in terms of the surface forces, it is necessary to solve the stress equations of equilibrium (7) and (8). The solutions must satisfy the boundary conditions of the applied forces and displacements. However, the equilibrium conditions of Equations 7 and 8 are not sufficient to determine the stresses.

The stress components are functions of the strain components and the stress components satisfy the four equations of compatibility (39). The equations of compatibility together with Equations 7 and 8 are necessary to furnish a sufficient number of equations to determine all stresses.

The stress components can be eliminated by the strain components expressed in terms of displacements by using Equations 1 and 2. Then, making a substitution for the stresses in the equilibrium equation a single partial differential equation can be formed.

Analogous to the corresponding theory of plane strain

$$\tau_{rz} = -\frac{\partial^2 \psi}{\partial r \partial z} \tag{40}$$

which when substituted in the equilibrium Equation 8 gives

$$-\frac{\partial^3 \psi}{\partial r^2 \partial z} + \frac{\partial \sigma_z}{\partial z} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial z} = 0 \tag{41}$$

Integration of Equation 41 with respect to z, gives

$$\sigma_z = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{42}$$

The equation does not include constants of integration as a function of r since ψ includes all such functions. Referring to Equation 1 we can write

$$\epsilon_r = \frac{\partial}{\partial r}(u) = \frac{\partial}{\partial r}(r\ \epsilon_\theta)$$

From Equations 3 we get

$$\sigma_r - \nu\ (\partial_\theta + \sigma_z) = \frac{\partial}{\partial r}\left\{\left[\sigma_\theta - \nu\ (\sigma_r + \sigma_z)\right]\ r\right\}$$

$$(1+\nu)(\sigma_r-\sigma_\theta) = r\ \frac{\partial}{\partial r}\left[\sigma_\theta-\nu\ (\sigma_r+\sigma_z)\right] \tag{43}$$

It has been found to be expedient to introduce a new function R defined by the equation

$$\sigma_r = \frac{\partial^2\psi}{\partial z^2} + R \tag{44}$$

substituting Equations 40, 44 and 43 in the equilibrium Equation 7 we obtain

$$\frac{\partial^3\psi}{\partial r\partial z^2} + \frac{\partial R}{\partial r} - \frac{\partial^3\psi}{\partial r\partial z^2} + \frac{1}{1+\nu}\ \frac{\partial}{\partial r}\left[\sigma_\theta - \nu\ (\sigma_r + \sigma_z)\right] = 0$$

$$(1+\nu)\ \frac{\partial R}{\partial r} + \frac{\partial}{\partial r}\left[\sigma_\theta - \nu\ (\sigma_r + \sigma_z)\right] = 0 \tag{45}$$

Let ψ include any arbitrary function of z such that

$$\sigma_\theta = \nu\nabla^2\psi - R \tag{46}$$

Also, let ψ be independent of θ such that

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\ \frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}$$

Adding Equations 42, 44 and 46, we get

$$\Phi = \sigma_r + \sigma_\theta + \sigma_z = (1+\nu)\,\nabla^2 \psi$$

Since Φ is a harmonic function (which can be shown by adding the three Equations of 21) we must have

$$\nabla^4 \psi = 0 \tag{47}$$

Substituting Equation 1 in 3 we get

$$u = \frac{r}{E}\left[\sigma_\theta - \nu\,(\sigma_r + \sigma_z)\right] \tag{48}$$

Using 42, 44 and 46 in the above equation we get

$$u = -\frac{r}{E}\,(1+\nu)\;R \tag{49}$$

Equation 4 gives

$$\tau_{rz} = \frac{E}{2(1+\nu)}\,\gamma_{rz}$$

Substituting Equations 2 and 40 in the above equation we write

$$-\frac{\partial^2\psi}{\partial r\partial z} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)$$

or

$$\frac{\partial w}{\partial r} = - \frac{2(1+\nu)}{E} \left(\frac{\partial^2 \psi}{\partial r \partial z}\right) - \frac{\partial u}{\partial z}$$

Differentiating Equation 49 with respect to z and substituting

$$\frac{\partial w}{\partial r} = - \frac{2(1+\nu)}{E} \left(\frac{\partial^2 \psi}{\partial r \partial z}\right) + \frac{r}{E} (1+\nu) \frac{\partial R}{\partial z} \qquad (50)$$

Substituting 1 in 3

$$\frac{\partial w}{\partial z} = \frac{1}{E}\left[\sigma_z - \nu \; (\sigma_r + \sigma_\theta) \right]$$

$$= \frac{1}{E}\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \nu \; \left(\frac{\partial^2 \psi}{\partial z^2} + \nu \nabla^2 \psi\right)\right]$$

$$= \frac{1}{E}\left[(1-\nu^2)\left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r}\right) - \nu \; (1+\nu) \frac{\partial^2 \psi}{\partial z^2}\right]$$

$$= \frac{1}{E}\left[(1-\nu^2) \nabla^2 \psi - (1+\nu) \frac{\partial^2 \psi}{\partial z^2}\right] \qquad (51)$$

Equations 50 and 51 are compatible only if

$$- \frac{2}{E} (1+\nu) \frac{\partial^3 \psi}{\partial r \partial z^2} + \frac{r}{E} (1+\nu) \frac{\partial^2 R}{\partial z^2} = \frac{(1+\nu)}{E} \left[(1-\nu) \frac{\partial}{\partial r} \nabla^2 \psi - \frac{\partial^3 \psi}{\partial r \partial z^2}\right]$$

which resulss from equating the derivative of Equation 50 with respect to z to the derivative of Equation 51 with respect to r. Simplification gives

$$r \frac{\partial^2 R}{\partial z^2} = (1-\nu) \frac{\partial}{\partial r} \nabla^2 \psi + \frac{\partial^3 \psi}{\partial r \partial z^2} \tag{52}$$

If we introduce a function Ω in 52 such that

$$rR = \frac{\partial \psi}{\partial r} + \frac{\partial \Omega}{\partial \partial} \tag{53}$$

then

$$\frac{\partial^2 \Omega}{\partial z^2} = (1-\nu) \nabla^2 \psi \tag{54}$$

Substituting derivative of R with respect to z, in 50 we can write

$$\frac{\partial w}{\partial r} = \left(\frac{1+\nu}{E}\right) \left[-2 \frac{\partial^2}{\partial r \partial z} + \frac{\partial 2}{\partial r \partial z} + \frac{\partial^2 \Omega}{\partial r \partial z}\right]$$

$$\frac{\partial w}{\partial r} = -\left(\frac{1+\nu}{E}\right) \frac{\partial^2}{\partial r \partial z} + \frac{(1+\nu)}{E} \frac{\partial^2 \Omega}{\partial r \partial z} \tag{55}$$

Also, by substituting 54 in 51

$$\frac{\partial w}{\partial z} = \frac{1}{E} \left[(1+\nu) \frac{\partial^2 \Omega}{\partial z^2} - (1+\nu) \frac{\partial^2 \psi}{\partial z^2}\right] \tag{56}$$

Integrating 55 with respect to r we get

$$w = -\frac{(1+\nu)}{E}\left[\frac{\partial\psi}{\partial z} - \frac{\partial\Omega}{\partial z}\right] \tag{57}$$

From 53 and 49 we get

$$u = -\frac{(1+\nu)}{E}\left[\frac{\partial\psi}{\partial r} + \frac{\partial\Omega}{\partial r}\right] \tag{58}$$

Ω can be shown to be a harmonic function by using Equations 57 and 58. Let

$$\Delta = \epsilon_r + \epsilon_\theta + \epsilon_z = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \tag{59}$$

then

$$\Delta = -\frac{(1+\nu)}{E}\left[\nabla^2\psi + \nabla^2\Omega - 2\,\frac{\partial^2\Omega}{\partial z^2}\right]$$

By substituting 54 in the above we get

$$\Delta = \frac{(1+\nu)}{E}\left[(1-2\nu)\,\nabla^2\psi - \nabla^2\Omega\right]$$

But, adding Equations 3

$$\Delta = \frac{(1-2\nu)}{E}\,\Phi$$

and by adding Equations 42, 44 and 46 and substituting for Φ

$$\Delta = \frac{1-2\nu}{E}\left[(1+\nu)\,\nabla^2\psi\right]$$

Therefore

$$\nabla^2\Omega = 0$$

Letting $x' = \psi + \Omega$ and substituting 53 in 44 we obtain

$$\sigma_r = \frac{\partial^2\psi}{\partial z^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial\Omega}{\partial r} = \nabla^2 x' - \frac{\partial^2\psi}{\partial r^2} - \frac{\partial\Omega}{\partial z^2} - \frac{\partial^2\Omega}{\partial r^2}$$

Using 54 in the above equation and noting $\nabla^2\Omega = 0$

$$\sigma_r = \nabla^2 x' - \frac{\partial^2 x'}{\partial r^2} - (1-\nu)\,\nabla^2\psi = \nu\,\nabla^2 x' - \frac{\partial^2 x'}{\partial r^2}$$

From Equations 46 and 42 we obtain

$$\sigma_\theta = \nu\,\nabla^2\psi - \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r}\frac{\partial\Omega}{\partial r} = \nu\,\nabla^2 x' - \frac{1}{r}\frac{\partial x'}{\partial r}$$

and

$$\sigma_z = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} = \nabla^2\psi - \frac{\partial^2\psi}{\partial z^2}$$

$$= \nabla^2\psi - \frac{\partial^2 x'}{\partial z^2} + (1-\nu)\,\nabla^2\psi$$

$$= (2-\nu)\,\nabla^2 x' - \frac{\partial^2 x'}{\partial z^2}$$

Setting $x' = \frac{\partial X_o}{\partial z}$ the above three equations for σ_r, σ_θ, σ_z can be written as

$$\sigma_r = \frac{\partial}{\partial z}\left[\nu\,\nabla^2 X_o - \frac{\partial^2 X_o}{\partial r^2}\right] \qquad (60)$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\nu\,\nabla^2 X_o - \frac{1}{r}\frac{\partial X_o}{\partial r}\right] \qquad (61)$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\nu)\,\nabla^2 X_o - \frac{\partial^2 X_o}{\partial z^2}\right] \tag{62}$$

Substituting these three equations in the equilibrium Equation 7 we obtain

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\,\nabla^2 X_o - \frac{\partial^2 X_o}{\partial z^2}\right] \tag{63}$$

Substituting these values of σ_r, σ_θ, σ_z and τ_{rz} in the second equilibrium Equation 8 results in

$$\nabla^4 X_o = 0 \tag{64}$$

Making use of the above values of σ_r, σ_θ, σ_z in Equations 1 and 3 we have

$$u = -\frac{(1+\nu)}{E}\,\frac{\partial^2 X_o}{\partial r \partial z} \tag{65}$$

and

$$w = \frac{(1+\nu)}{E}\left[(1-2\nu)\;\nabla^2 X_o + \frac{\partial^2 X_o}{\partial r^2} + \frac{1}{r}\,\frac{\partial X_o}{\partial r}\right] \tag{66}$$

The problem of determining the distribution of stress in a solid of revolution now becomes that of finding a solution to the biharmonic Equation 64 and satisfying certain boundary conditions. Once the function X_o is determined, Equations 60 to 63 will give the stresses and 65 and 66 the displacements.

For convenience we introduce a new variable "X" defined as being equal to KX_o, where K is a constant. The new variable will not affect the logic developed to this point and the form of Equation 64 remains the same. The other Equations 60 to 65 become functions of X and must satisfy Equation 67 below.

$$\nabla^4 X = 0 \tag{67}$$

If $K = \frac{1}{2(\lambda+\mu)}$ and $X = KX_o$, Equations 60, 61, 62, 63, 65 and 66 become

$$\sigma_r = \frac{\partial}{\partial r}\left[\lambda\nabla^2 X - 2\ (\lambda + \mu)\ \frac{\partial^2 X}{\partial r^2}\right] \tag{68}$$

$$\sigma_\theta = \frac{\partial}{\partial r}\left[\lambda\nabla^2 X - \frac{2}{r}\ (\lambda + \mu)\ \frac{\partial^2 X}{\partial r^2}\right] \tag{69}$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(3\lambda + 4\mu)\ \nabla^2 X - 2\ (\lambda + \mu)\ \frac{\partial^2 X}{\partial z^2}\right] \tag{70}$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(\lambda + 2\mu)\ \nabla^2 X - 2\ (\lambda + \mu)\ \frac{\partial^2 X}{\partial z^2}\right] \tag{71}$$

$$u = -\ \frac{(\lambda + \mu)}{\mu}\ \frac{\partial^2 x}{\partial r \partial z} \tag{72}$$

$$w = \frac{(\lambda + 2\mu)}{\mu}\nabla^2 X - (\frac{\lambda + \mu)}{\mu}\ \frac{\partial^2 X}{\partial z^2 \partial r} \tag{73}$$

Boundary and Interface Conditions

Assumptions

For the analysis of the layered system it is assumed that there is continuous surface of contact between the layers at all times, in addition to the assumption of isotropy and homogeneity of all layers.

Since in actual practice the different layers are neither welded nor are frictionless at the interfaces, it is assumed that there is some relative displacement of the different layers and that this is accomplished by overcoming the friction developed at the interfaces.

Physical significance of the assumptions

At the upper surface where z is equal to zero the shear stress between the tire and the pavement is assumed to be zero and the tire contact pressure itself is assumed to be a parabolic distribution.

The equilibrium conditions and the continuity of the material through the different layers gives us the continuity of the vertical stress σ_z, shear stress τ_{rz} and vertical displacement w through the interface. Based on the assumption of the existence of movement and friction at the interfaces, and according to the experiments conducted by Terzaghi and Peck (24), the shear stress τ_{rz} developed at the interface is proportional to the movement between the layers.

As z approaches infinity all stresses and displacements approach zero.

Boundary and interface conditions of the problem

The above discussion gives us the boundary conditions which must be imposed on the partial differential Equation 67. These conditions are mathematically stated below:

(1) At $z = 0$ $\qquad \tau_{rz} = 0 \qquad (74)$

(2) At $z = 0$ $\qquad \sigma_z = p\left[1 - (\frac{r}{a})^2\right]$ for $0<r<a \qquad (75)$

$\sigma_z = 0$ for $a<r<\infty$

(3) At any interface $\sigma_{z_j} = \sigma_{z_{j+1}} \qquad (76)$

(4) At any interface $w_j = w_{j+1} \qquad (77)$

(5) At any interface $\tau_{rz_j} = \tau_{rz_{j+1}} \qquad (78)$

(6) At any interface $\tau_{rz_j} = \beta\ (U_j - U_{j+1}) \qquad (79)$

(7) As $z \longrightarrow \infty \qquad X \longrightarrow 0 \qquad (80)$

SOLUTION OF THE PROBLEM

The determination of stresses and displacements now lies in finding a solution, satisfying the boundary conditions (74) to (80), to the biharmonic Equation 67.

The biharmonic equation of two variables r and z can be transformed to an ordinary second order differential equation by means of the Hankel transform.

The boundary conditions must also be changed to this form so that all equations contain derivatives with respect to z only, instead of r and z.

Reduction of the Biharmonic Equation

The Hankel transform $F_n(\xi)$ is defined so that if a function f(r) satisfies certain requisite conditions then

$$F_n(\xi) = \int_0^{\infty} f(r)\ J_n(\xi r)\ r\ dr$$

and

$$f(r) = \int_0^{\infty} F_n(\xi)\ J_n\ (\xi r)\xi\ \ d\xi$$

by the inversion theorem.

Now for the partial differential equation $\nabla^4 X = 0$ in polar co-ordinates we have

$$\nabla^4 X = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) X = 0 \qquad (81)$$

Multiplying both sides of (81) by $J_o(\xi r)$ r and integrating with respect to r from zero to infinity we obtain

$$\int_o^{\infty} \left(\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial z^2}\right) r\, J_o(\xi r)\, dr = 0$$

where

$$\eta = \frac{\partial^2 X}{\partial r^2} + \frac{1}{r}\frac{\partial X}{\partial r} + \frac{\partial^2 X}{\partial z^2}$$

denoting the left hand side by L

$$L = \frac{\partial^2}{\partial z^2}\int_o^{\infty} \eta\, J_o(\xi r)\, r\, dr + \int_o^{\infty} \frac{\partial}{\partial r}\left[r\frac{\partial \eta}{\partial r}\right] J_o(\xi r)\, dr$$

$$L = \frac{\partial^2 \bar{\eta}}{\partial z^2} + \left[r\frac{\partial \eta}{\partial r} J_o(\xi r)\right]_o^{\infty} - \int_o^{\infty} r\frac{\partial \eta}{\partial r}\frac{\partial}{\partial r}\left[J_o(\xi r)\right] dr$$

where

$$\bar{\eta} = \int_o^{\infty} \eta\; J_o\,(\xi r)\, r\, dr$$

Since $\frac{\partial \eta}{\partial r}$ is assumed to be finite for all values of r

$$\left[r\frac{\partial \eta}{\partial r} J_o(\xi r)\right]_o^{\infty} = 0$$

Also since

$$\frac{\partial}{\partial r} J_o(r) = - J_1(r)$$

$$\frac{\partial}{\partial r}\left[J_o(\xi r)\right] = -\xi\left[J_1(\xi r)\right]$$

$$\frac{\partial}{\partial r}\left[r\, J_1(\xi r)\right] = (\xi r)\quad J_o(\xi r)$$

Using the above identities L can be further simplified as shown below.

$$L = \frac{\partial^2 \bar{\eta}}{\partial z^2} - \int_0^\infty r \frac{\partial \eta}{\partial r} \left[- \xi J_1(\xi r) \right] \quad dr$$

$$L = \frac{\partial^2 \bar{\eta}}{\partial z^2} + \xi \int_0^\infty r \frac{\partial \eta}{\partial r} \; J_1(\xi r) \; dr$$

$$= \frac{\partial^2 \bar{\eta}}{\partial z^2} + \left[\eta \xi \; J_1(\xi r) \right]_0^\infty - \xi \int_0^\infty \eta \frac{\partial}{\partial r} \left[r \; J_1(\xi r) \right] dr$$

$$= \frac{\partial^2 \bar{\eta}}{\partial z^2} - \xi^2 \int_0^\infty \eta \; J_o(\xi r) \; r \; dr$$

$$= \left(\frac{\partial^2}{\partial z^2} - \xi^2\right) \int_0^\infty \eta \; J_o(\xi r) \; r \; dr$$

$$= \left(\frac{\partial^2}{\partial z^2} - \xi^2\right) \int_0^\infty \left(\frac{\partial^2 X}{\partial r^2} + \frac{1}{r} \frac{\partial X}{\partial r} + \frac{\partial^2 X}{\partial z^2}\right) r \; J_o(\xi r) \; dr$$

Repeating the operation once again we obtain

$$\int_0^\infty r \; \nabla^4 X \; J_o(\xi r) \; dr = 0 = \left(\frac{\partial^2}{\partial z^2} - \xi^2\right)^2 \int_0^\infty r \; X \; J_o(\xi r) \; dr$$

If we use $G(\xi, z)$ for the zero-order Hankel transform of X we get

$$G(\xi, z) = \int_0^\infty r \; X \; J_o(\xi r) \; dr \qquad (82)$$

$$\left(\frac{d^2}{dz^2} - \xi^2\right)^2 \; G(\xi, z) = 0$$

Solving this differential equation we obtain

$$G(\xi, z) = (A + B\,z)\, e^{-\xi z} + (C+Dz)\, e^{\xi z} \qquad (83)$$

The constants A, B, C and D can be determined by imposing boundary conditions (74) to (80).

Hankel Transforms of the Stresses and Displacements

The boundary conditions 74 to 80 must also be transformed by means of the Hankel transform. This requires the transformation of the stress and displacement Equations 68 to 73.

<u>Vertical stress</u>

$$\sigma_z = \frac{\partial}{\partial z}\left[(3\lambda+4\mu)\,\nabla^2 X - 2(\lambda+\mu)\,\frac{\partial^2 X}{\partial z^2}\right]$$

Multiplying both sides by $r\,J_o(\xi r)$ and integrating with respect to r from zero to ∞

$$\int_o^\infty r\,\sigma_z\,J_o(\xi r)\,dr = (3\lambda+4\mu)\,\frac{\partial}{\partial z}\int_o^\infty r\,\nabla^2 X\,J_o(\xi r)\,dr$$

$$- 2(\lambda+\mu)\,\frac{\partial}{\partial z}\int_o^\infty r\,\frac{\partial^2 X}{\partial z^2}\,J_o(\xi r)\,dr$$

If $\bar{\sigma}_z$ is the Hankel transform of σ_z since

$$\int_o^\infty r\,\nabla^2 X\,J_o(\xi r)\,dr = \left(\frac{\partial^2}{\partial z^2} - \xi^2\right)\int_o^\infty r\,X\,J_o(\xi r)\,dr$$

$$= \left(\frac{\partial^2}{\partial z^2} - \xi^2\right) G$$

$$\bar{\sigma}_z = (3\lambda+4\mu)\left(\frac{\partial^3}{\partial z^3} - \xi^2 \frac{\partial}{\partial z}\right) G - 2(\lambda+\mu)\frac{\partial^3 G}{\partial z^3}$$

$$= (\lambda+2\mu)\frac{\partial^3 G}{\partial z^3} - (3\lambda+4\mu)\,\xi^2 \frac{\partial G}{\partial z} \tag{84}$$

Therefore by the inversion theorem

$$\sigma_z = \int_0^\infty \xi \left[(\lambda+2\mu)\frac{\partial^3 G}{\partial z^3} - (3\lambda+4\mu)\,\xi^2 \frac{\partial^3 G}{\partial z^3}\right] J_o(\xi r)\, d\xi \tag{85}$$

Shear stress

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(\lambda+2\mu)\nabla^2 X - 2(\lambda+\mu)\frac{\partial^2 X}{\partial z^2}\right]$$

$$\int_0^\infty r\,\tau_{rz} J_1(\xi r)\, dr = (\lambda+2\mu)\int_0^\infty r \frac{\partial}{\partial r}\nabla^2 X\, J_1(\xi r)\, dr$$

$$- 2(\lambda+\mu)\int_0^\infty r \frac{\partial^3 X}{\partial z^2 \partial r} J_1(\xi r)\, dr$$

$$\bar{\tau}_{rz} = (\lambda+2\mu)\left\{\left[r\, J_1(\xi r)\nabla^2 X\right]_0^\infty - \int_0^\infty \nabla^2 X \frac{\partial}{\partial r}\left[r\, J_1(\xi r)\right] dr\right\}$$

$$- 2(\lambda+\mu)\frac{\partial^2}{\partial z^2}\int_0^\infty r \frac{\partial X}{\partial r} J_1(\xi r)\, dr$$

$$= -(\lambda+2\mu)\,\xi\int_0^\infty \nabla^2 X\, r\, J_o(\xi r)\, dr + 2(\lambda+\mu)\frac{\partial^2}{\partial z^2}$$

$$\left\{\left[\xi\int_0^\infty r\, X\, J_o(\xi r)\, dr\right] - \left[X\, r\, J_1(\xi r)\right]_0^\infty\right\}$$

$$= -(\lambda+2\mu)\,\xi\left(\frac{\partial^2}{\partial z^2} - \xi^2\right) G + 2(\lambda+\mu)\,\xi\,\frac{\partial^2 G}{\partial z^2}$$

$$= \lambda\,\xi\,\frac{\partial^2 G}{\partial z^2} + (\lambda+2\mu)\,\xi^3 G \tag{86}$$

By inversion

$$\tau_{rz} = \int_0^{\infty} \xi^2 \left[\lambda \frac{\partial^2 G}{\partial z^2} + (\lambda+2\mu)\xi^2 G \right] J_1(\xi r)\, d\xi \qquad (87)$$

Vertical displacement

$$w = \frac{\lambda+2\mu}{\mu} \nabla^2 X - \frac{\lambda+\mu}{\mu} \frac{\partial^2 X}{\partial z^2}$$

$$\int_0^{\infty} r\, w\, J_o(\xi r)\, dr = \frac{\lambda+2\mu}{\mu} \int_0^{\infty} r \nabla^2 X\, J_o(\xi r)\, dr$$

$$- \frac{\lambda+\mu}{\mu} \int_0^{\infty} r \frac{\partial^2 X}{\partial z^2} J_o(\xi r)\, dr$$

$$\bar{w} = \frac{\lambda+2\mu}{\mu} \left(\frac{\partial^2}{\partial z^2} - \xi^2\right) \int_0^{\infty} r\, X\, J_o(\xi r)\, dr$$

$$- \frac{\lambda+\mu}{\mu} \frac{\partial^2}{\partial z^2} \int_0^{\infty} r\, X\, J_o(\xi r)\, dr$$

$$= \frac{\lambda+\mu}{\mu} \left(\frac{\partial^2}{\partial z^2} - \xi^2\right) G - \frac{\lambda+\mu}{\mu} \frac{\partial^2 G}{\partial z^2}$$

$$= \frac{\partial^2 G}{\partial z^2} - \frac{\lambda+2\mu}{\mu} \xi^2 G \qquad (88)$$

By inversion

$$w = \int_0^{\infty} \xi \left[\frac{\partial^2 G}{\partial z^2} - \frac{\lambda+2\mu}{\mu} \xi^2 G \right] J_o(\xi r)\, d\xi \qquad (89)$$

Horizontal displacement

$$u = - \frac{\lambda+\mu}{\mu} \frac{\partial^2 X}{\partial z \partial r}$$

$$\int_0^{\infty} r\ u\ J_1(\xi r)\ dr = -\frac{\lambda+\mu}{\mu}\frac{\partial}{\partial z}\int_0^{\infty}\frac{\partial X}{\partial r}\ r\ J_1(\xi r)\ dr$$

$$\bar{u} = -\frac{\lambda+\mu}{\mu}\left\{\left[X\ r\ J_1(\xi r)\right]_0^{\infty} - \frac{\partial}{\partial r}\int_0^{\infty} X\ \frac{\partial}{\partial r}\left[r\ J_1(\xi r)\right]\right\}\ dr$$

$$= +\frac{\lambda+\mu}{\mu}\ \xi\ \frac{\partial}{\partial z}\int_0^{\infty} X\ r\ J_0(\xi r)\ dr$$

$$= +\frac{\lambda+\mu}{\mu}\ \xi\ \frac{\partial G}{\partial z} \tag{90}$$

By inversion

$$u = \frac{\lambda+\mu}{\mu}\int_0^{\infty}\xi^2\ \frac{\partial G}{\partial z}\ J_1(\xi r)\ d\xi \tag{91}$$

For a multi-layered media by Equation 83, the transform of X for the jth layer can be written as

$$G_j = (A_j + B_j\ z)\ e^{-\xi z} + (C_j + D_j\ z)\ e^{\xi z} \tag{92}$$

This function must satisfy the boundary conditions at the interfaces. Using the approach just developed we obtain the transforms of stresses and displacements for a jth layer as follows:

$$(\bar{\sigma}_z)_j = (\lambda_j+2\mu_j)\ \frac{\partial^3 G_j}{\partial z^3} - (3\lambda_j+4\mu_j)\ \xi^2\ \frac{\partial G_j}{\partial z} \tag{93}$$

$$(\sigma_z)_j = \int_0^{\infty}\xi\left[(\lambda_j+2\mu_j)\ \frac{\partial^3 G_j}{\partial z^3} - (3\lambda_j+4\mu_j)\ \xi^2\ \frac{\partial G_j}{\partial z}\right] J_0(\xi r)\ d\xi \tag{94}$$

$$(\bar{\tau}_{rz})_j = \lambda_j \ \xi \frac{\partial^2 G_j}{\partial z^3} + (\lambda_j + 2\mu_j) \ \xi^3 \ G_j \tag{95}$$

$$(\bar{\tau}_{rz})_j = \int_o^{\infty} \xi^2 \left[\lambda_j \frac{\partial^2 G_j}{\partial z^2} + (\lambda_j + 2\mu_j) \xi^2 \ G_j \right] J_1(\xi r) \ d\xi \tag{96}$$

$$(\bar{w})_j = \frac{\partial^3 G_j}{\partial z^2} - \frac{\lambda_j + 2\mu_j}{\mu_j} \ \xi^2 \ G_j \tag{97}$$

$$(w)_j = \int_o^{\infty} \xi \left[\frac{\partial^3 G_j}{\partial z^2} - \frac{\lambda_j + 2\mu_j}{\mu_j} \ \xi^2 \ G_j \right] \ J_o(\xi r) \ d\xi \tag{98}$$

$$(\bar{u})_j = \frac{\lambda_j + \mu_j}{\mu_j} \ \xi \ \frac{\partial G_j}{\partial z} \tag{99}$$

$$(\bar{u})_j = \left(\frac{\lambda_j + \mu_j}{\mu_j}\right) \int_o^{\infty} \xi^2 \ \frac{\partial G_j}{\partial z} \ J_1(\xi r) \ d\xi \tag{100}$$

The boundary conditions transform to the following form:

(1) At $z = 0$ $\quad (\bar{\tau}_{rz})_1 = 0$ (101)

(2) At $z = 0$ $\quad (\bar{\sigma}_z)_1 = F(\xi)$ (102)

where $F(\xi)$ is the Hankel transform of $\sigma_z = P \left[1-(r/a)^2 \right]$ for $0<r<0$ and $\sigma_z = 0$ for $a<r<\infty$

(3) At any interface $(\bar{\sigma}_z)_j = (\bar{\sigma}_z)_{j+1}$ (103)

(4) At any interface $(\bar{w})_j = (\bar{w})_{j+1}$ (104)

(5) At any interface $(\bar{\tau}_{rz})_j = (\bar{\tau}_{rz})_{j+1}$ (105)

(6) At any interface $(\bar{\tau}_{zr})_j = \beta \left[(\bar{u})_j - (\bar{u})_{j+1} \right]$ (106)

(7) At $z \longrightarrow \infty$ $\qquad G_n \longrightarrow 0$ (107)

where n represents the number of layers.

Determination of the Constants of Integration

By applying boundary condition (101) to Equation 95 we obtain

$$\lambda_1 \frac{\partial^2 G_1}{\partial z^2} + (\lambda_1 + 2\mu_1)\xi^2 G_1 = 0 \text{ at } z = 0$$

The function G and its derivatives will now be written in terms of A, B, C and D for use in the developments that follow.

$$G_j = (A_j + B_j z)e^{-\xi z} + (C_j + D_j z)e^{\xi z}$$

$$\frac{dG_j}{dz} = (-A_j\xi + B_j - B_j\xi z)e^{-\xi z} + (C_j\xi + D_j + D_j z\xi)e^{\xi z}$$

$$\left(\frac{dG_j}{dz}\right)_{z=0} = -A_1\xi + B_1 + C_1\xi + D_1$$

$$\frac{d^2G_j}{dz^2} = (A_j\xi - 2B_j + B_j\xi z)\xi e^{-\xi z} + (C_j\xi + 2D_j - D_j z\xi)\xi e^{\xi z}$$

$$\left(\frac{d^3G_j}{dz^2}\right)_{z=0} = (A_1\xi - 2B_1 + C_1\xi + 2D_1)\xi$$

$$\frac{d^3G_j}{dz^3} = (-A_j\xi + 3B_j - B_j\xi z)\xi^2 e^{-\xi z} + (C_j\xi + 3D_j - D_j\xi z)\xi^2 e^{\xi z}$$

$$\left(\frac{d^3G_j}{dz^3}\right)_{z=0} = (-A_1\xi + 3B_1 + C_1\xi - 3D_1)\xi^2$$

$$\lambda_1 \; (A_1 \xi^2 - 2B_1 \xi + C_1 \xi^2 + 2D_1 \xi) + (\lambda_1 + 2\mu_1) \xi^2 (A_1 + C_1) = 0$$

or

$$(A_1 + C_1) \; (\lambda_1 + \mu_1 \xi) = \lambda_1 \; (B_1 - D_1) \tag{108}$$

For the first layer using boundary condition (102) and Equation 93,

$$F(\xi) = (\lambda_1 + 2\mu_1) \frac{\partial^3 G_1}{\partial z^2} - (3\lambda_1 + 4\mu_1) \xi^2 \frac{\partial G_1}{\partial z} \text{ at } z = 0$$

$$= 2\xi^3 (A_1 - C_1)(\lambda_1 + \mu_1) + 2\xi^2 \mu_1 (B_1 + D_1) \tag{109}$$

At any interface at a depth H, interface condition (103) applied to Equation 93 for any two layers gives

$$(\lambda_j + 2\mu_j) \frac{\partial^3 G_j}{\partial z^3} - (3\lambda_j + 4\mu_j) \xi^2 \frac{\partial G_j}{\partial z} = (\lambda_{j+1} + 2\mu_{j+1}) \frac{\partial^2 G_{j+1}}{\partial z^3}$$

$$- (3\lambda_{j+1} + 4\mu_{j+1}) \xi^2 \frac{\partial G_{j+1}}{\partial z^3}$$

$$A_j e^{-\xi H} \xi^3 (2\lambda_j + 2\mu_j) + 2B_j \xi^2 e^{-\xi H} \left[\mu_j + \xi H (\lambda_j + \mu_j) \right]$$

$$- C_j e^{\xi H} \xi^3 (2\lambda_j + 2\mu_j) + 2 \; D_j \xi^2 e^{\xi H} \left[\mu_j - \xi H (\lambda_j + \mu_j) \right] =$$

$$A_{j+1} e^{-\xi H} \xi^3 (2\lambda_{j+1} + 2\mu_{j+1}) + 2B_{j+1} \xi^2 e^{-\xi H} \left[\mu_{j+1} + \xi H (\lambda_{j+1} + \mu_{j+1}) \right]$$

(110)

$$\mp C_{j+1} e^{\xi H} \xi^3 (2\lambda_{j+1} + 2\mu_{j+1}) + 2D_{j+1} \xi^2 e^{\xi H} \left[\mu_{j+1} - \xi H (\lambda_{j+1} + \mu_{j+1}) \right]$$

At an interface H, (104) applied to Equation 97 gives

$$\frac{\partial^2 G_j}{\partial z^2} - \frac{\lambda_j + 2\mu_j}{\mu_j}\xi^2 G_j = \frac{\partial^2 G_{j+1}}{\partial z^2} - \frac{\lambda_{j+1}+2\mu_{j+1}}{\mu_{j+1}}\xi^2 G_{j+1}$$

$$A_j\, e^{-\xi H}\xi^2 \left(-\frac{\lambda_j+\mu_j}{\mu_j}\right) - B_j\, e^{-\xi H}\xi\left[2 + \frac{H\xi}{\mu_j}(\lambda_j+\mu_j)\right]$$

$$+ C_j\, e^{\xi H}\xi^2 \left(-\frac{\lambda_j+\mu_j}{\mu_j}\right) - D_j\, e^{\xi H}\xi\left[-2 + \frac{H\xi}{\mu_j}(\lambda_j+\mu_j)\right] =$$

$$A_{j+1}\, e^{-\xi H}\xi^2 \left(-\frac{\lambda_{j+1}+\mu_{j+1}}{\mu_{j+1}}\right)$$

$$- B_{j+1}\, e^{-\xi H}\xi\left[2 + \frac{H\xi}{\mu_{j+1}}(\lambda_{j+1}+\mu_{j+1})\right]$$

$$+ C_{j+1}\, e^{\xi H}\xi^2 \left(-\frac{\lambda_{j+1}+\mu_{j+1}}{\mu_{j+1}}\right)$$

$$- D_{j+1}\, e^{\xi H}\xi\left[-2 + \frac{H\xi}{\mu_{j+1}}(\lambda_{j+1}+\mu_{j+1})\right] \qquad (111)$$

By (105) applied to Equation 95 we get

$$\lambda_j \frac{\partial^2 G_j}{\partial z^2} + (\lambda_j + 2\mu_j)\,\xi^2 G_j = \lambda_{j+1}\frac{\partial^2 G_{j+1}}{\partial z^2} + (\lambda_{j+1}+2\mu_{j+1})\,\xi^2 G_{j+1}$$

Then proceeding as before

$$A_j \xi^2 e^{-\xi H}(\lambda_j+\mu_j) + B_j \xi e^{-\xi H}\left[-\lambda_j + \xi H(\lambda_j + \mu_j)\right]$$

$$+ C_j \xi^2 e^{\xi H}(\lambda_j+\mu_j) + D_j \xi e^{\xi H}\left[\lambda_j + \xi H(\lambda_j + \mu_j)\right] =$$

$$A_{j+1}\xi^2 e^{-\xi H}(\lambda_{j+1}+\mu_{j+1})+B_{j+1}\xi e^{-\xi H}\left[-\lambda_{j+1}+\xi H(\lambda_{j+1}+\mu_{j+1})\right]$$

$$+C_{j+1}\xi^2 e^{\xi H}(\lambda_{j+1}+\mu_{j+1})+D_{j+1}\xi e^{\xi H}\left[\lambda_{j+1}+\xi H(\lambda_{j+1}+\mu_{j+1})\right] \quad (112)$$

condition (106) applied to Equations 95 and 99 for two layers gives

$$\lambda_j \frac{\partial^2 G_j}{\partial z^2} + (\lambda_j+2\mu_j)\xi^2 G_j = \beta\left[\frac{\lambda_j+\mu_j}{\mu_j}\frac{\partial G_j}{\partial z} - \frac{\lambda_{j+1}+\mu_{j+1}}{\mu_{j+1}}\frac{\partial G_{j+1}}{\partial z}\right]$$

$$A_j\xi e^{-\xi H}\left[(\lambda_j+\mu_j)(2\xi+\frac{\beta}{\mu_j})\right] + B_j e^{-\xi H}\left[-2\lambda_j\xi+2\xi^2 H(\lambda_j+\mu_j)\right.$$

$$\left.-\frac{\beta}{\mu_j}(\lambda_j+\mu_j)(1-\xi H)\right] + C_j\xi e^{\xi H}\left[(\lambda_j+\mu_j)(2\xi-\frac{\beta}{\mu_j})\right]$$

$$+D_j e^{\xi H}\left[2\lambda_j\xi+2\xi^2 H(\lambda_j+\mu_j) - \frac{\beta}{\mu_j}(\lambda_j+\mu_j)(1+\xi H)\right] =$$

$$A_{j+1}\xi e^{-\xi H}\left[-\frac{\beta}{\mu_{j+1}}(\lambda_{j+1}+\mu_{j+1})\right]$$

$$+B_{j+1}e^{-\xi H}\left[\frac{\beta}{\mu_{j+1}}(\lambda_{j+1}+\mu_{j+1})(1-\xi H)\right]$$

$$+C_{j+1}\xi e^{\xi H}\left[\frac{\beta}{\mu_{j+1}}(\lambda_{j+1}+\mu_{j+1})\right]$$

$$+D_{j+1}e^{\xi H}\left[\frac{\beta}{\mu_{j+1}}(\lambda_{j+1}+\mu_{j+1})(1+\xi H)\right] \quad (113)$$

For the nth layer using boundary condition (107) and Equation 92

$$(A_n+B_n\ z)\ e^{-\xi z} + (C_n+D_n\ z)\ e^{+\xi z} \longrightarrow 0 \text{ as } z \longrightarrow \infty$$

This gives

$$C_n = 0 \tag{114}$$

and

$$D_n = 0 \tag{115}$$

For a system consisting of n layers we have (4 n-2) constants to be evaluated. The topmost layer gives 2 condition equations 108 and 109, and the (n-1) interfaces between n layers give 4(n-1) equations similar to 110 to 113. These conditions furnish sufficient equations to solve for A_j, B_j, C_j and D_j.

Computation Procedure

The (4n-2) equations were written in the matrix notation in the form $QY_o=S$ where Q is the $\left[(4n-2)(4n-2)\right]$ matrix containing the coefficients of the constants A_j, B_j, C_j and D_j. Y_o is a $\left[(4n-2)\ (1)\right]$ matrix containing the terms A_j, B_j, C_j, D_j's, and S is a $\left[(4n-2)\ (1)\right]$ matrix containing the right hand sides of all the equations. In this matrix all terms except one are zero. These matrices are presented in the following pages.

These matrices were solved to obtain the values of the constants A_j, B_j, C_j and D_j, which were used in Equations 94 and 98 to obtain the values of stresses and displacements. The equations were integrated over the interval 0 to ∞ with respect to for different values of z to get the effects at different depths.

$$
\begin{bmatrix}
\mathrm{xxxxxx} & & & & \\
\mathrm{xxxxxx} & & & & \\
\mathrm{xxxxxx} & & & & \\
\mathrm{xxxxxx} & & & & \\
\mathrm{xxxxxx} & & & & \\
\mathrm{xxxxxx} & & & & \\
\mathrm{xxxxxx} & & & & \\
\mathrm{xxxxxx} & & & & \\
 & \cdots\cdots & & & \\
 & \cdots\cdots & & & \\
 & \cdots\cdots & & & \\
 & & \mathrm{xxxxxx} & & \\
 & & \mathrm{xxxxxx} & & \\
 & & \mathrm{xxxxxx} & & \\
 & & \mathrm{xxxxxx} & & \\
 & & & \cdots\cdots & \\
 & & & \cdots\cdots & \\
 & & & \cdots\cdots & \\
 & & & & \mathrm{xxxxxx} \\
 & & & & \mathrm{xxxxxx}
\end{bmatrix}
\begin{bmatrix}
A_1 \\ B_1 \\ C_1 \\ D_1 \\ A_2 \\ B_2 \\ C_2 \\ D_2 \\ \cdot \\ \cdot \\ \cdot \\ A_j \\ B_j \\ C_j \\ D_j \\ \cdot \\ \cdot \\ \cdot \\ A_n \\ B_n
\end{bmatrix}
=
\begin{bmatrix}
0 \\ \frac{8}{3\xi^5 E_1}\left[2\, J_1(\xi) - J_o(\xi)\right] \\ \\ 0 \\ 0 \\ 0 \\ \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \\ \\
\end{bmatrix}
$$

$$[Q] \quad [Y_o] = [S]$$

Matrix Q, Y_o and S for parabolic load conditions.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	
$+3\xi$	0	0	0	-2	0	0	(1)
3ξ	0	0	0	$+1$	0	0	(2)
$3\xi e^{-\xi H_1}$	$-3\xi e^{-\xi H_1}\dfrac{E_2}{E_1}$	0	0	$e^{-\xi_1}(1+3\xi H_1)$	$-e^{-\xi H_1}(1+3\xi H_1)\dfrac{E_2}{E_1}$	0	(3)
0	$3\xi e^{-\xi H_2}\dfrac{E_2}{E_1}$	$-3\xi e^{-\xi H_2}\dfrac{E_3}{E_1}$	0	0	$e^{-\xi H_2}(1+3\xi H_2)\dfrac{E_2}{E_1}$	$-e^{-\xi H_2}(1+3\xi H_2)\dfrac{E_3}{E_1}$	(4)
0	0	$3\xi e^{-\xi H_3}\dfrac{E_3}{E_1}$	$-3\xi e^{-\xi H_3}\dfrac{E_4}{E_1}$	0	0	$e^{-\xi H_3}(1+3\xi H_3)\dfrac{E_3}{E_1}$	(5)
$-3\xi e^{-\xi H_1}$	$+3\xi e^{-\xi H_1}$	0	0	$-e^{-\xi H_1}(2+3\xi H_1)$	$e^{-\xi H_1}(2+3\xi H_1)$	0	(6)
0	$-3\xi e^{-\xi H_2}$	$+3\xi e^{-\xi H_2}$	0	0	$-e^{-\xi H_2}(2+3\xi H_2)$	$e^{-\xi H_2}(2+3\xi H_2)$	(7)

Table 1a. Matrix Q for a four-layered system

(8)	(9)	(10)	(11)	(12)	(13)	(14)	
0	$+3\xi$	0	0	$+2$	0	0	(1)
0	-3ξ	0	0	$+1$	0	0	(2)
0	$-3\xi e^{+\xi H_1}$	$+3\xi e^{+\xi H_1}$ $\frac{E_2}{E_1}$	0	$(-1+3\xi H_1)\ -e^{\xi H_1}$	$(-1+3\xi H_1)\ e^{\xi H_1}\ \frac{E_2}{E_1}$	0	(3)
0	0	$-3\xi e^{+\xi H_2}$ $\frac{E_2}{E_1}$	$3\xi e^{+\xi H_2}$ $\frac{E_3}{E_1}$	0	$(-1+3\xi H_2)\ -e^{\xi H_2}\ \frac{E_2}{E_1}$	$(-1+3\xi H_2)\ e^{\xi H_2}\ \frac{E_3}{E_1}$	(4)
$(1+3\xi H_3)\ -e^{-\xi H_3}\ \frac{E_4}{E_1}$	0	0	$-3\xi e^{+3H_3}$ $\frac{E_3}{E_1}$	0	0	$(-1+3\xi H_3)\ -e^{\xi H_3}\ \frac{E_3}{E_1}$	(5)
0	$-3\xi e^{+\xi H_1}$	$+3\xi e^{+\xi H_1}$	0	$(-2+3\xi H_1)\ -e^{+\xi H_1}$	$(-2+3\xi H_1)\ e^{\xi H_1}$	0	(6)
0	0	$-3\xi e^{+\xi H_2}$	$3\xi e^{+\xi H_2}$	0	$(-2+3\xi H_2)\ -e^{\xi H_2}$	$(-2+3\xi H_2)\ e^{\xi H_2}$	(7)

Table 1b. Matrix Q for a four-layered system

(1)	(2)	(3)	(4)	(5)	(6)	(7)	
$e^{-\xi H_3}(2+3\xi H_3)$	0	0	$-3\xi e^{4\xi H_3}$	0	0	$-e^{\xi H_3}(-2+3\xi H_3)$	(8)
0	$3\xi e^{\xi H_1}$	$-3\xi e^{\xi H_1}\frac{E_2}{E_1}$	0	$e^{\xi H_1}(2+3\xi H_1)$	$-e^{\xi H_1}(2+3\xi H_1)\frac{E_2}{E_1}$	0	(9)
0	0	$3\xi e^{\xi H_2}\frac{E_2}{E_1}$	$-3\xi e^{\xi H_2}\frac{E_3}{E_1}$	0	$e^{\xi H_2}(2+3\xi H_2)\frac{E_2}{E_1}$	$-e^{\xi H_2}(2+3\xi H_2)\frac{E_3}{E_1}$	(10)
$-e^{-\xi H_3}(-2+3\xi H_3)\frac{E_4}{E_1}$	0	0	$3\xi e^{\xi H_3}\frac{E_3}{E_1}$	0	0	$e^{\xi H_3}(2+3\xi H_3)\frac{E_3}{E_1}$	(11)
0	$\xi e^{\xi H_1}(\frac{3}{4}\xi-\frac{\beta_1}{E_1})$	$\xi e^{\xi H_1}\frac{\beta_1}{E_1}$	0	$e^{\xi H_1}\left[\frac{\xi}{2}+\frac{3}{4}\xi H_1-\frac{\beta_1}{E_1}(1+\xi H_1)\right]$	$e^{\xi H_1}\frac{\beta_1}{E_1}(1+\xi H_1)$	0	(12)
0	0	$\xi e^{\xi H_2}(\frac{3}{4}\xi-\frac{\beta_2}{E_2})\frac{E_2}{E_1}$	$\xi e^{\xi H_2}\frac{\beta_2}{E_1}$	0	$e^{\xi H_2}\left[\frac{\xi}{2}+\frac{3}{4}\xi^2 H_2-\frac{\beta_2}{E_2}(1+\xi H_2)\right]\frac{E_2}{E_2}$	$e^{\xi H_2}\frac{\beta_2}{E_1}(1+\xi H_2)$	(13)
$e^{-\xi H_3}\frac{\beta_3}{E_1}(1-\xi H_3)$	0	0	$\xi e^{\xi H_3}(\frac{3}{4}\xi+\frac{\beta_3}{E_3})\frac{E_3}{E_1}$	0	0	$e^{\xi H_3}\left[\frac{\xi}{2}+\frac{3}{4}\xi^2 H_3-\frac{\beta_3}{E_3}(1+\xi H)\right]\frac{E_3}{E_1}$	(14)

Table 1c. Matrix Q for a four-layered system

	(8)	(9)	(10)	(11)	(12)	(13)	(14)
(8)	0	0	$-3\xi e^{\xi H_3}$	$+3\xi e^{\xi H_3}$	0	0	$-e^{-\xi H_3}(2+3\xi H_3)$
(9)	$3\xi e^{-3H_1}$	$-3\xi e^{-\xi H_1}\frac{E_2}{E_1}$	0	0	$e^{-\xi H_1}(-2+3\xi H_1)$	$-e^{-\xi H_1}(-2+3\xi H_1)\frac{E_2}{E_1}$	0
(10)	0	$3\xi e^{-\xi H_2}\frac{E_2}{E_1}$	$-3\xi e^{-\xi H_2}\frac{E_3}{E_1}$	0	0	$e^{-\xi H_2}(-2+3\xi H_2)\frac{E_2}{E_1}$	$-e^{-\xi H_2}(-2+3\xi H_2)\frac{E_3}{E_1}$
(11)	0	0	$3\xi e^{-\xi H_3}\frac{E_3}{E_1}$	$-3\xi e^{-\xi H_3}\frac{E_4}{E_1}$	0	0	$e^{-\xi H_3}(-2+3\xi H_3)\frac{E_3}{E_1}$
(12)	$\xi e^{-\xi H_1}(\frac{3}{4}\xi+\frac{O_1}{E_1})$	$-\xi e^{-\xi H_1}\frac{O_1}{E_1}$	0	0	$e^{-\xi H_1}[-\frac{\xi}{2}+\frac{3}{4}\xi^2 H_1 \quad \frac{O_1}{E_1}(1-\xi H_1)]$	$e^{-\xi H_1}\frac{O_1}{E_1}(1-\xi H_1)$	0
(13)	0	$\xi e^{-\xi H_2}(\frac{3}{4}\xi+\frac{O_2}{E_2})\frac{E_2}{E_1}$	$-\xi e^{-\xi H_2}\frac{O_2}{E_1}$	0	0	$e^{-\xi H_2}[-\frac{\xi}{2}+\frac{3}{4}\xi^2 H_2 - \frac{O_2}{E_2}(1-\xi H_2)]\frac{E_2}{E_1}$	$e^{-\xi H_2}\frac{O_2}{E_1}(1-\xi H_2)$
(14)	0	0	$\xi e^{-\xi H_3}(\frac{3}{4}\xi+\frac{O_3}{E_3})\frac{E_3}{E_1}$	$-\xi e^{-\xi H_3}\frac{O_3}{E_1}$	0	0	$e^{-\xi H_3}[-\frac{\xi}{2}+\frac{3}{4}\xi^2 H_3 - \frac{O_3}{E_3}(1-\xi H_3)]\frac{E_3}{E_1}$

Table 1d. Matrix Q for a four-layered system

COMPUTER PROGRAM

Evaluation of Integrals

The integral expressions of (94) and (98) representing the stresses and displacements are evaluated numerically, at different depths in the layered medium, by using the Gaussian quadrature formula (27) as explained below. The numerical evaluation was done by the use of an "I.B.M. 7074" computer.

The range of integration is divided into N intervals and the s point Gaussian formula is applied to each interval. If

$$I_T = \int_0^{\infty} I(\xi) \quad J_n(\xi r) \, d\xi$$

then I_T can be written as

$$I_T = \sum_{k=1}^{N} \left[\sum_{i=1}^{s} W_i \, I(i, k) \, J_n \, (i, k) \right] \tag{116}$$

where $I(i, k)$ and J_n (i, k) represent the values of I and J_n at i^{th} subinterval of k^{th} interval. W_i is the weighting coefficient of the Gaussian formula. The inner sum of i gives the value of the integral for one interval and the outer sum gives the value for the N intervals.

The intervals were chosen to fall between points where the Bessel function J_n that appears in the integral is zero. The number of intervals N is infinite since the range of integration is from zero to infinity but the numerical program uses only a finite number of intervals. The integration process is continued until the absolute value of the integral in the given interval is less than some specified percent of the sum obtained to this point.

The program was developed for a parabolic distribution of load distributed over a circular area. Four layers were assumed and Poisson's ratio ν for the material was assumed to be 1/3. Investigations into the properties of granular road materials and bituminous materials have shown that the Poisson's ratio probably lies between 0.3 and 0.4. (28).

The vertical stress values σ_z and vertical displacement values w were evaluated only for the point of symmetry since maximum values exist on the line of symmetry. The shear stress τ_{rz} and the radial displacement u on the line of symmetry x are zero.

The non-dimensional parameters that are to be specified in using the program are the ratios of the thicknesses of the different layers to the radius of the loaded area and the ratios of elastic moduli of all layers to the elastic modulus of the first layer.

The time required to compute the vertical effects at eleven different depths for a four layer system is nearly 3 minutes. The time of computation depends upon the relative elastic moduli and the ratios of thicknesses of different layers.

This program can be,used with relatively minor modifications for different types of load patterns and different frictional coefficients between layers.

Outline of the Program

The computer program is based on the sequence given below.

As described in the previous paragraphs the integration from zero to infinity is evaluated as a double summation. The procedure is explained with reference to the case of τ_{rz}.

$$\tau_{rz} = \sum_{k=1}^{N} \left[\sum_{i=1}^{s} W_i \ \tau(i,k) \ J_1 \ (i,k) \right]$$

The inner sum is computed as follows:

1. First the limits of the k^{th} interval are set. In the case of τ_{rz} they are the consecutive values for which $J_1(\xi r)$ are zero. But since the value of r for which τ_{rz} are significant are 1/2, 1 and 2, etc., the consecutive zero's of $J_1(\xi)$ can be used. Choosing the zero's of $J_1(\xi)$ instead of $J_1(\xi r)$ will not have a bad effect on the evaluation of τ_{rz} for $r = 1/2$ but the accuracy at $r = 2$ will be slightly less. For the vertical effects at $r = 0$, consecutive ξ's for which $J_0(\xi)$ is zero are used. In both cases the first ξ is taken to be zero. These units of the

interval are all obtained from a subroutine of the "Integration Routine" in which the zero's of $J_1(\xi)$ and $J_o(\xi)$ are specified as constants.

2. The values of ξ for the i^{th} subintervals are evaluated by using Gaussian constants. In the present case, 6 points are calculated.

3. Evaluation of $\tau(i,k)$

a. Matrices Q, Y_o and S are formed for the first ξ value. It should be noted that Q was written in a slightly different form for the use of the computer.

b. The matrix is inverted by a standard library inversion routine and we get the values of A_j, B_j, C_j and D_j's for a certain value of ξ.

c. All the derivatives of G_j's are formed for this ξ and the corresponding z values using the formula given on page 59.

d. The integrand of τ_{rz} given in (96) is formed and multiplied by W(i), the weighting constant of the Gaussian formula.

e. Items a, b, c, and d are repeated s times for the succeeding ξ values within the interval and added together. This sum is multiplied by $d\xi$ obtained from the Gaussian formula.

Repeating steps 2, 3a, b, d, c and e, for consecutive intervals of $J_1(\xi r)$ and adding we obtain τ_{rz}. This cycle is repeated until the absolute value of the inner sum or the area under the graph of the integrand for the last interval is less than a specified percent of the total sum of areas obtained from all preceding intervals up to the final interval. The specified percent of tolerance in this program is one percent, i.e., the area of the last interval must be equal to or less than one hundredth of the sum of the areas of all the preceding intervals.

The basic steps are given in the form of a chart shown in Figure 5.

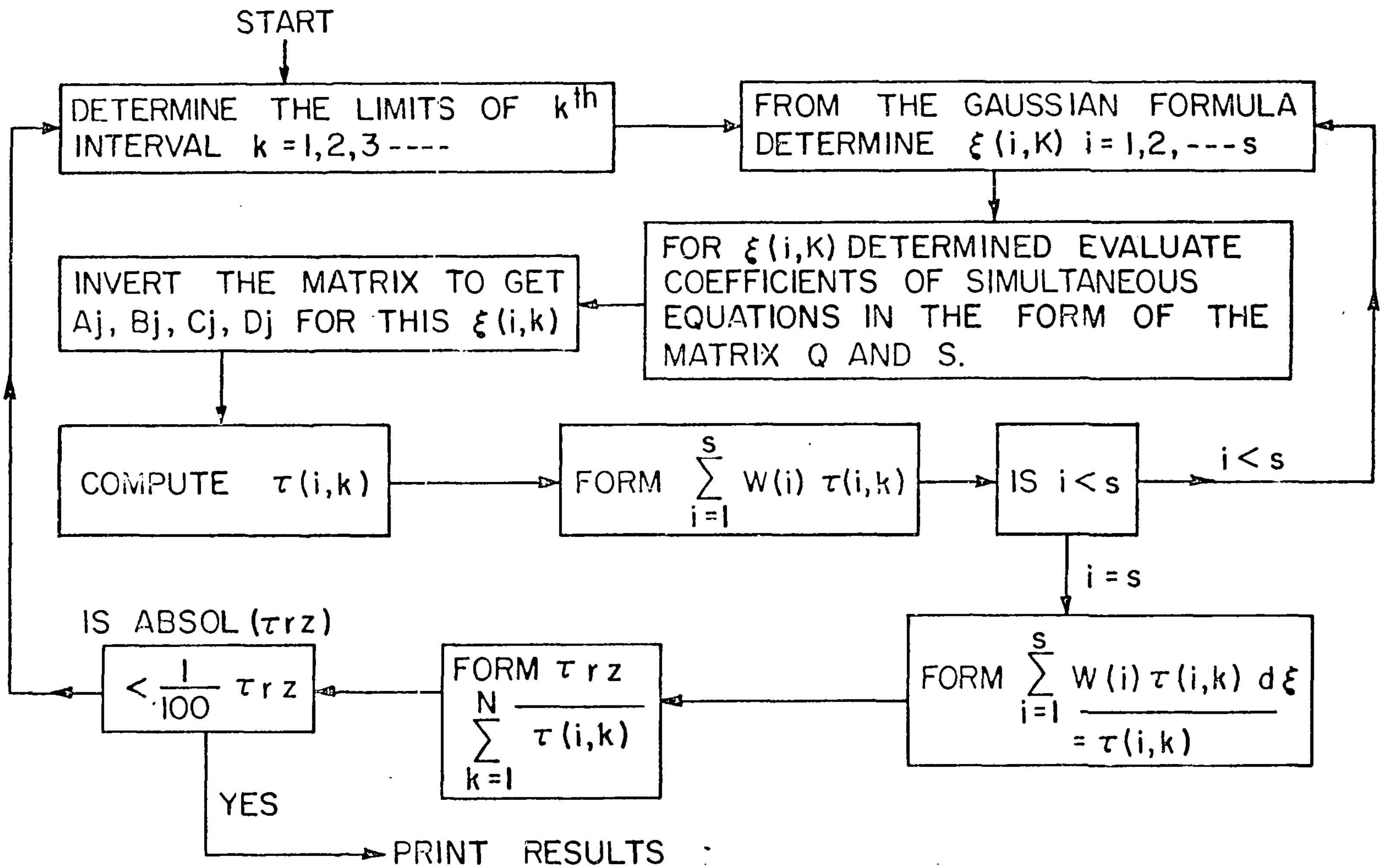

Figure 5. Outline of the computer program

RESULTS AND GRAPHS

Numerical values of normal stresses and vertical displacements have been evaluated for the following standard pavements:

Tire imprint radius	8"	9"		12"	
	(1)	(1)	(2)	(1)	(2)
Surface course thickness	2"	3"	3"	4"	6"
Base course thickness	6"	6"	9"	8"	12"
Subbase thickness	12"	18"	12"	16"	18"

The effects on the pavements have been evaluated both for a silty subgrade and gravel subgrade. The elastic moduli have been assumed to be: 5,000,000 psi, 1,000,000 psi, 100,000 psi and 10,000 psifor surface course, base course, subbase and silty subgrade respectively. For gravel subgrade the moduli are taken as 5,000,000 psi, 500,000 psi, 100,000 psi and 500,000 psi for the consecutive layers. The values of β are assumed to be tan 75^{o} for the first interface and tan 85^{o} for the second and third interfaces, since it is generally thought that there is less bonding between the wearing surface and the base than for lower interfaces.

The actual stresses at depth z can be obtained by entering the graph of the corresponding tire imprint size at z/a and multiplying the stress influence coefficient by P. The deflection is obtained by multiplying the deflection factor by $(\frac{Pa}{E_1})$, and the units will be the same as that of a, since $\frac{P}{E_1}$ is dimensionless.

Vertical stress influence coefficient $\sigma_{z/p}$

Tire imprint radius = 8 inches

$\beta_1 = 3.73$	$\beta_2 = 11.43$	$\beta_3 = 11.43$
$H_1 = 0.25$	$H_2 = 1.00$	$H_3 = 2.50$
$h_1 = 2"$	$h_2 = 6"$	$h_3 = 12"$

z/a	Gravel subgrade	Silty subgrade
0.00	1.000	1.000
0.06	0.266	0.266
0.12	0.889	0.888
0.19	0.814	0.814
0.25	0.784	0.786
0.50	0.704	0.620
0.75	0.529	0.306
1.00	0.346	0.187
1.75	0.152	0.186
2.50	0.195	0.036
3.50	0.118	0.232

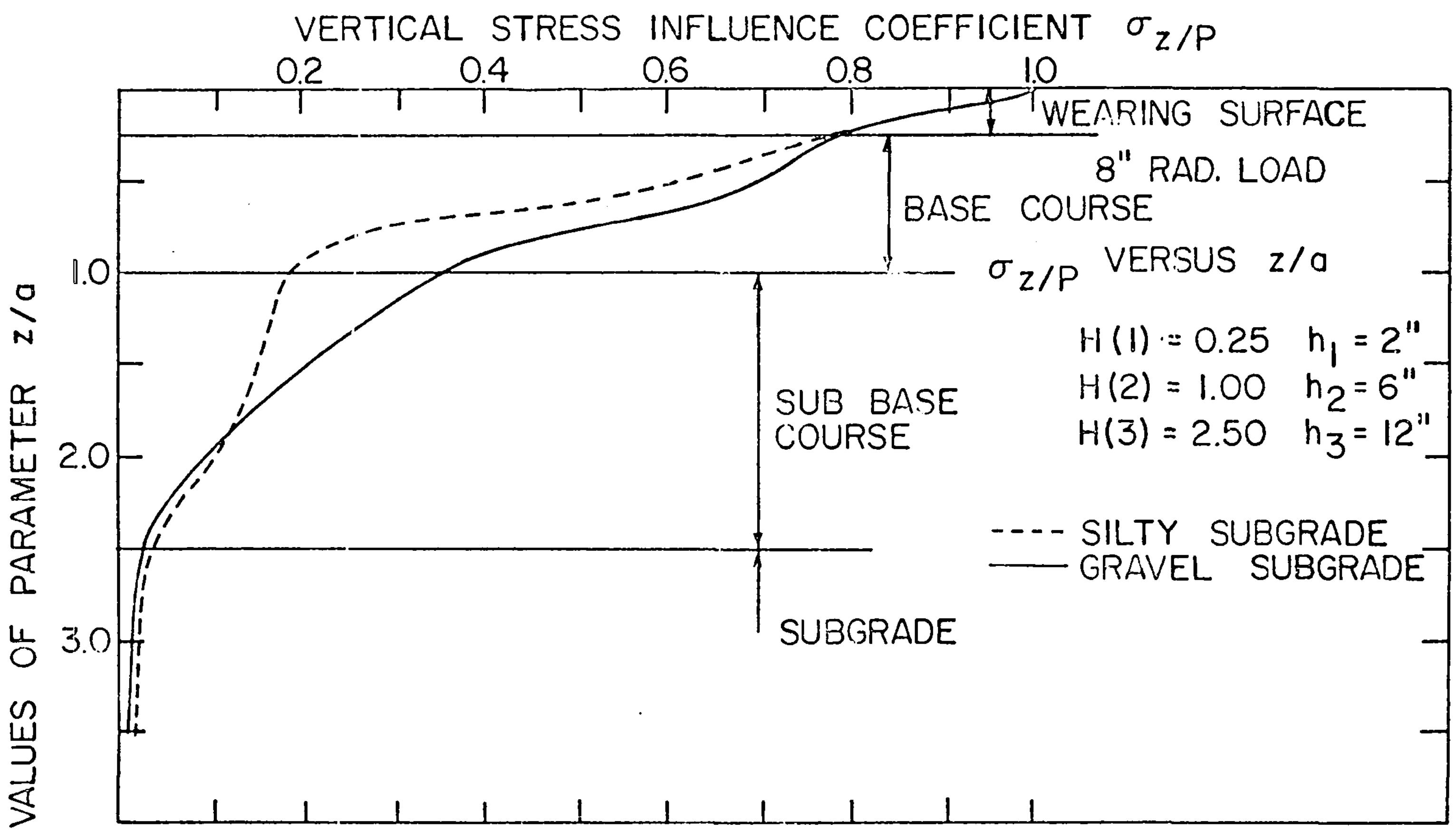

Figure 6. Vertical stress influence coefficient versus z/a

Vertical deflection factor $w(\frac{E_1}{p} \times \frac{1}{a})$

Tire imprint radius = 8 inches

$\beta_1 = 3.73$	$\beta_2 = 11.43$	$\beta_3 = 11.43$
$H_1 = 0.25$	$H_2 = 1.00$	$H_3 = 2.50$
$h_1 = 2"$	$h_2 = 6"$	$h_3 = 12"$

z/a	Gravel subgrade	Silty subgrade
0.00	16.70	97.28
0.06	16.67	97.25
0.12	16.65	97.22
0.19	16.58	97.09
0.25	16.44	96.88
0.50	15.62	96.50
0.75	14.61	96.25
1.00	13.79	96.00
1.75	3.20	90.06
2.50	2.04	83.84
3.50	1.80	68.48

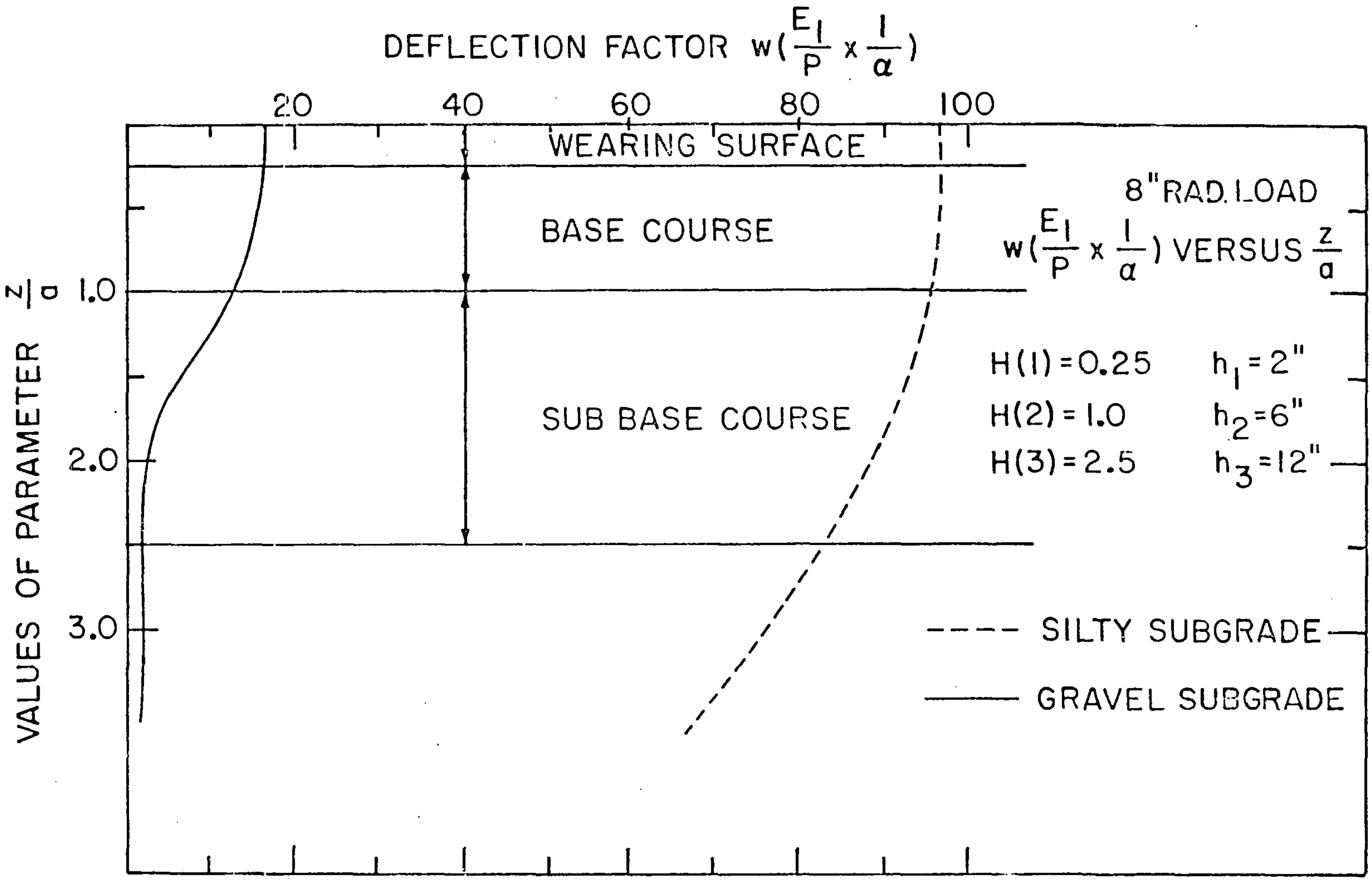

Figure 7. Deflection factor versus z/a

Vertical stress influence coefficient, $\sigma_{z/p}$

Tire imprint radius = 9 inches

$\beta_1 = 3.73$	$\beta_2 = 11.43$	$\beta_3 = 11.43$
$H_1 = 0.33$	$H_2 = 1.00$	$H_3 = 3.00$
$h_1 = 3"$	$h_2 = 6"$	$h_3 = 18"$

z/a	Gravel subgrade	Silty subgrade
0.00	1.000	1.000
0.08	0.896	0.240
0.17	0.670	0.809
0.25	0.447	0.681
0.33	0.352	0.629
0.55	0.292	0.532
0.78	0.253	0.382
1.00	0.223	0.222
2.00	0.135	0.0811
3.00	0.063	0.236
4.00	0.015	0.171

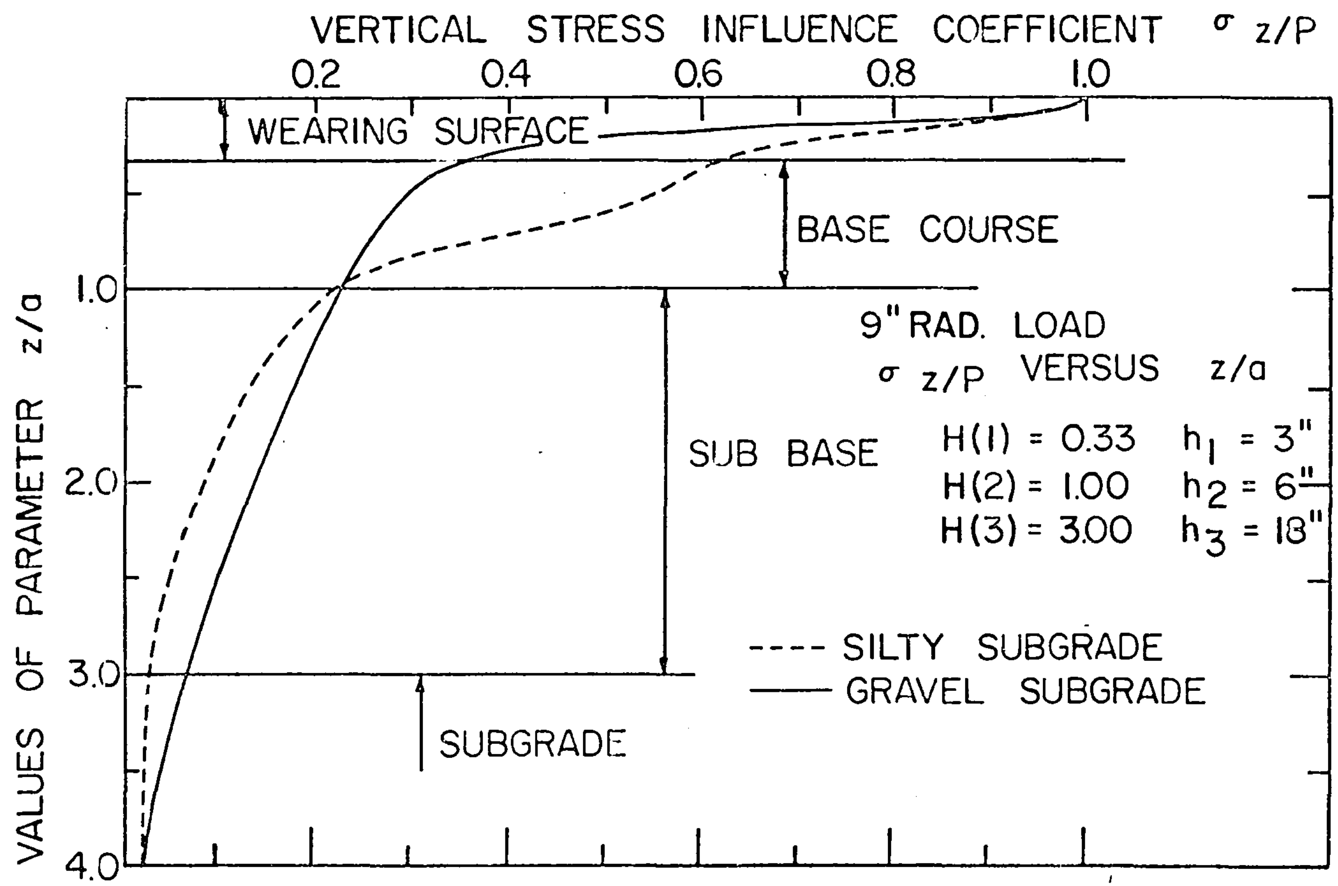

Figure 8. Vertical stress influence coefficient versus z/a

Vertical deflection factor $w(\frac{E_1}{p} \times \frac{1}{a})$

Tire imprint radius = 9 inches

$\beta_1 = 3.73$ $\beta_2 = 11.43$ $\beta_3 = 11.43$

$H_1 = 0.33$ $H_2 = 1.00$ $H_3 = 3.00$

$h_1 = 3"$ $h_2 = 6"$ $h_3 = 18"$

z/a	Gravel subgrade	Silty subgrade
0.00	20.19	22.05
0.08	20.18	82.00
0.17	20.07	81.99
0.25	20.06	81.86
0.33	20.04	81.62
0.55	20.00	81.50
0.78	19.70	81.22
1.00	18.47	80.06
2.00	7.91	75.73
3.00	2.19	67.54
4.00	1.70	56.29

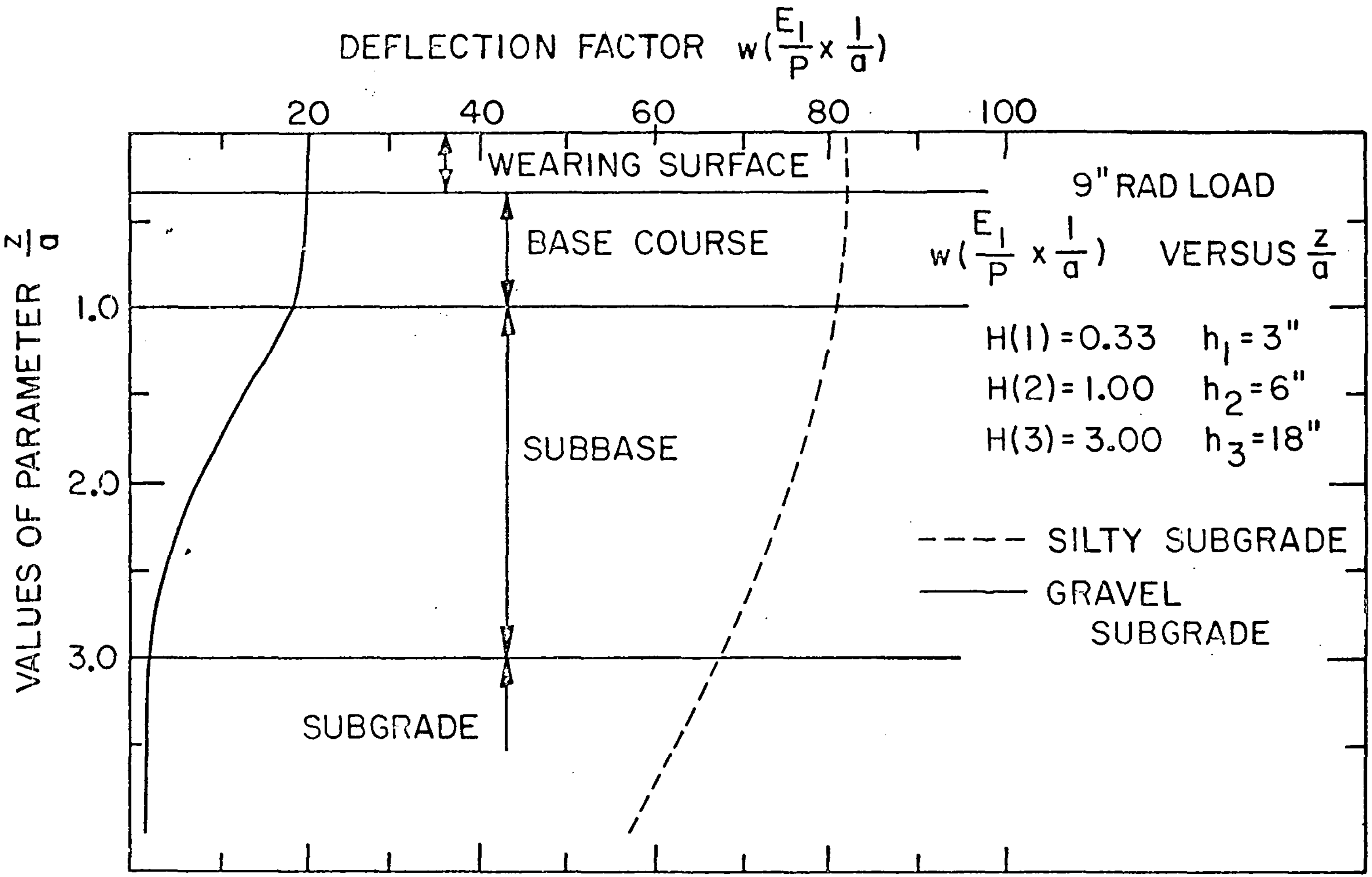

Figure 9, Deflection factor versus z/a

Vertical stress influence coefficient σ_z/p

Tire imprint radius = 9 inches

$\beta_1 = 3.73$	$\beta_2 = 11.43$	$\beta_3 = 11.43$
$H_1 = 0.33$	$H_2 = 1.33$	$H_3 = 2.66$
$h_1 = 3"$	$h_2 = 9"$	$h_3 = 12"$

z/a	Gravel subgrade	Silty subgrade
0.00	1.000	1.000
0.08	0.946	0.956
0.17	0.827	0.850
0.25	0.710	0.767
0.33	0.660	0.730
0.66	0.576	0.578
1.00	0.417	0.303
1.33	0.249	0.143
2.00	0.103	0.070
2.67	0.033	0.028
3.67	0.012	0.019

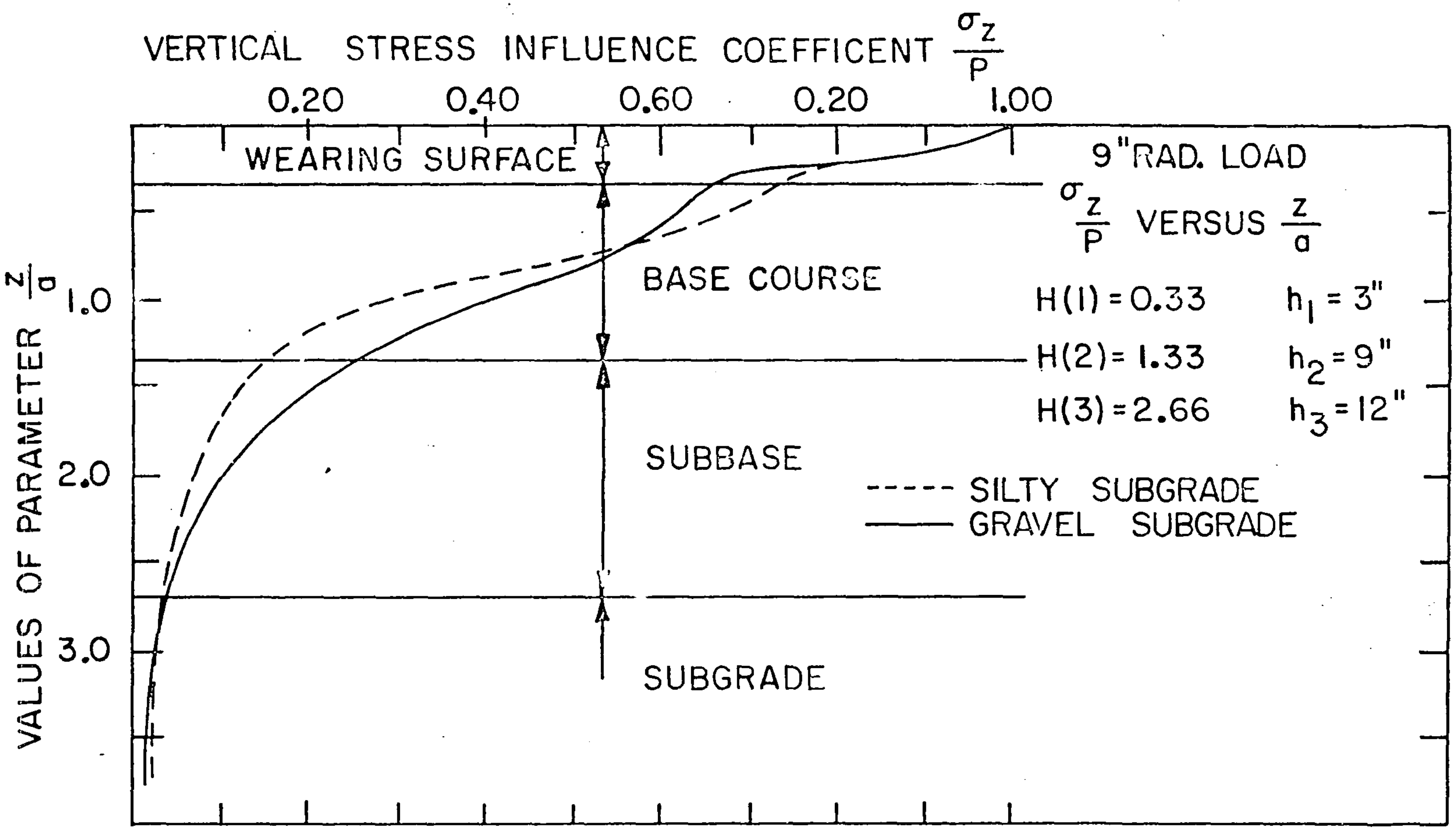

Figure 10. Vertical stress influence coefficient versus z/a

Vertical deflection factor $w(\frac{E_1}{p} \times \frac{1}{a})$

Tire imprint radius = 9 inches

$\beta_1 = 3.73$ $\beta_2 = 11.43$ $\beta_3 = 11.43$

$H_1 = 0.33$ $H_2 = 1.33$ $H_3 = 2.66$

$h_1 = 3"$ $h_2 = 9"$ $h_3 = 12"$

z/a	Gravel Subgrade	Silty subgrade
0.00	13.91	84.60
0.08	13.88	84.58
0.17	13.87	84.45
0.25	13.74	84.45
0.33	13.60	84.20
0.66	12.54	84.00
1.00	11.33	83.0
1.33	10.45	79.26
2.00	3.42	74.07
2.66	2.28	61.75
3.67	1.92	

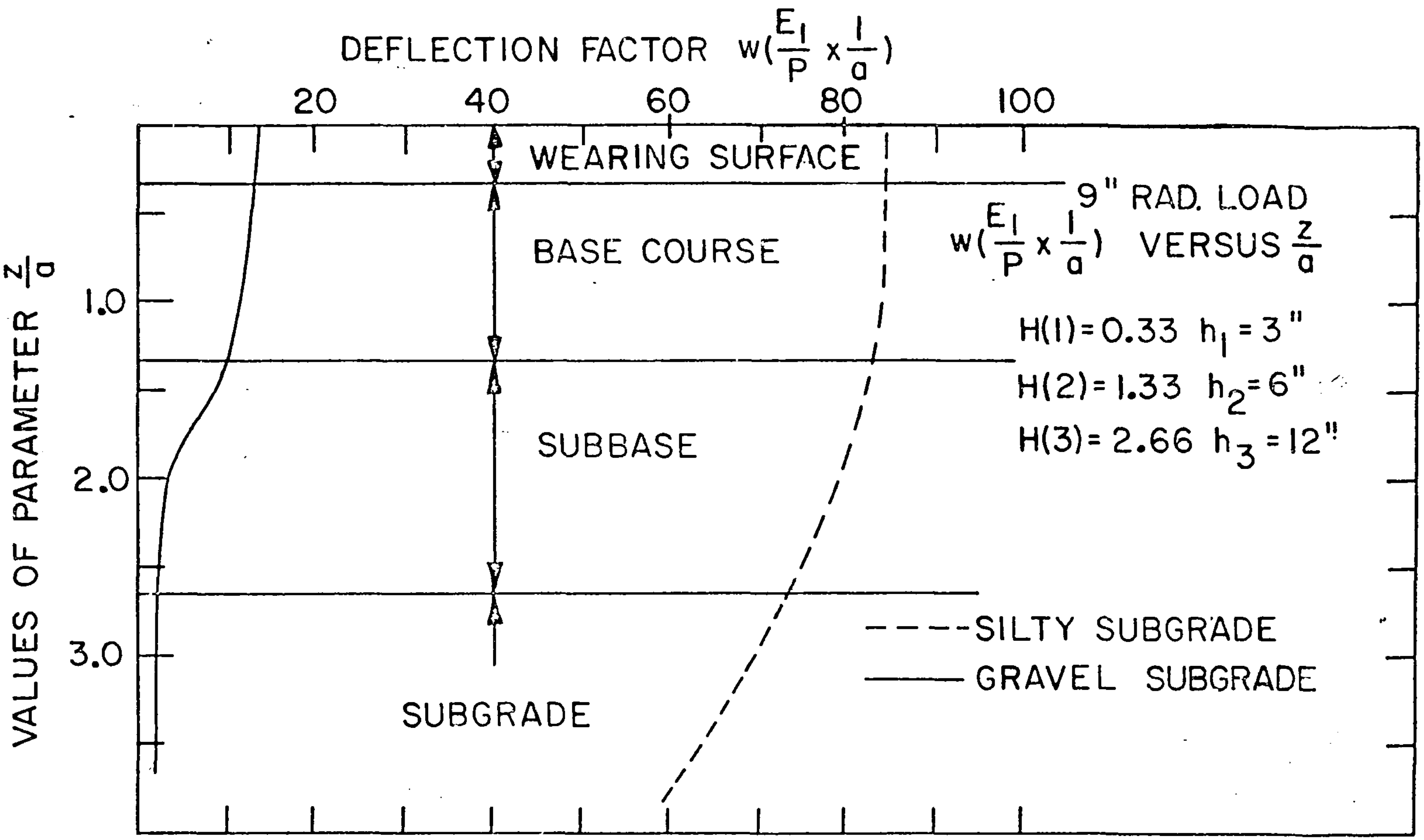

Figure 11. Deflection coefficient versus z/a

Vertical stress influence coefficient $\sigma_{z/p}$

Tire imprint radius = 12 inches

$\beta_1 = 3.73$	$\beta_2 = 11.43$	$\beta_3 = 11.43$
$H_1 = 0.33$	$H_2 = 1.00$	$H_3 = 2.33$
$h_1 = 4"$	$h_2 = 8"$	$h_3 = 16"$

z/a	Gravel subgrade	Silty subgrade
0.00	1.000	1.000
0.08	0.942	0.931
0.17	0.815	0.781
0.25	0.630	0.635
0.33	0.637	0.577
0.55	0.588	0.472
0.78	0.465	0.250
1.00	0.325	0.184
1.67	0.144	0.143
2.33	0.042	0.047
3.33	0.008	0.030

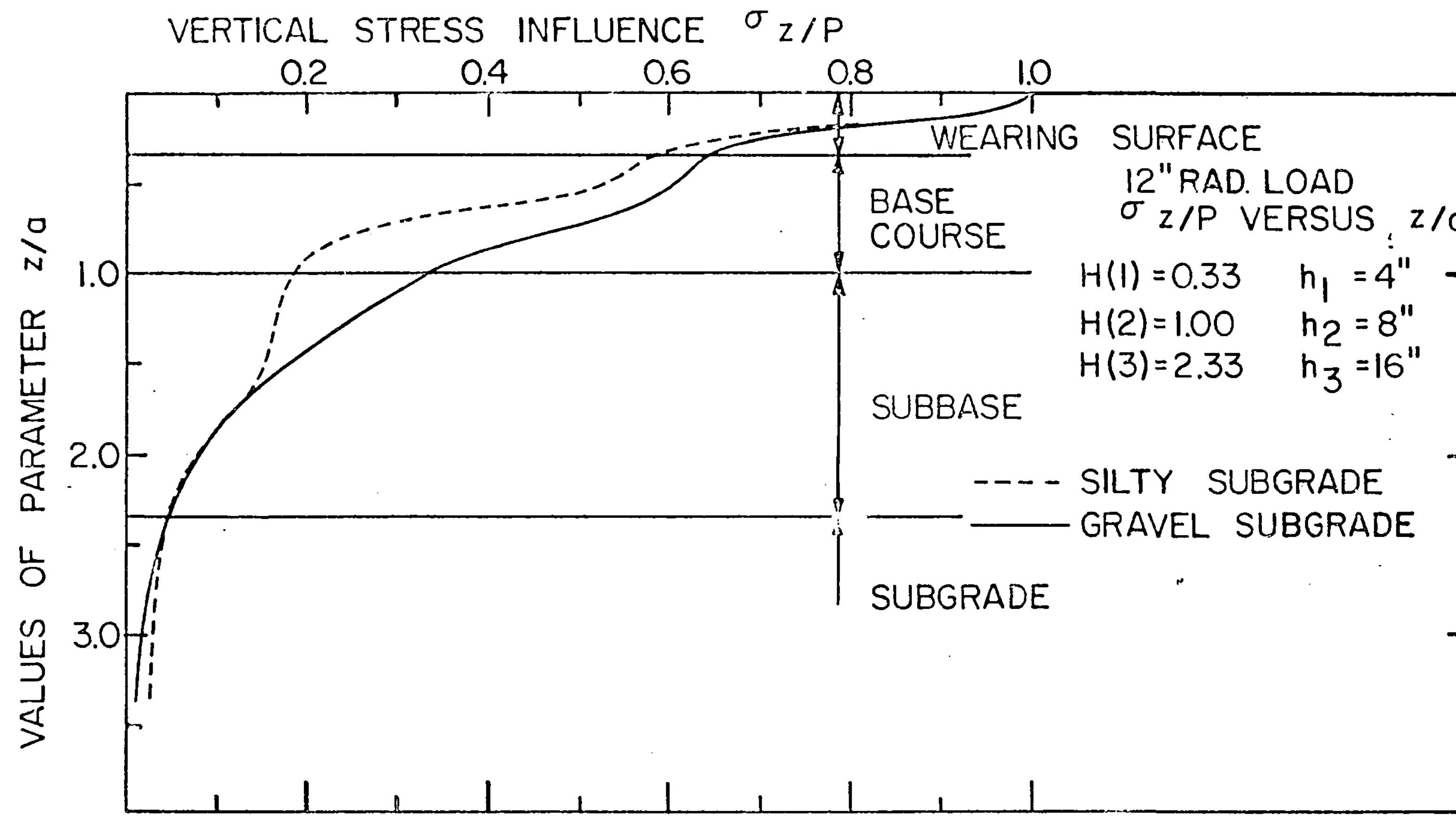

Figure 12. Vertical stress influence coefficient versus z/a

Vertical deflection factor $w(\frac{E_1}{p} \times \frac{1}{a})$

Tire imprint radius = 12 inches

$\beta_1 = 3.73$	$\beta_2 = 11.43$	$\beta_3 = 11.43$
$H_1 = 0.33$	$H_2 = 1.00$	$H_3 = 2.33$
$h_1 = 4"$	$h_2 = 8"$	$h_3 = 16"$

z/a	Gravel subgrade	Silty subgrade
0.00	15.04	111.56
0.08	15.01	111.51
0.17	14.99	111.48
0.25	14.30	111.33
0.33	14.72	111.03
0.55	14.21	111.00
0.75	13.52	111.00
1.00	12.92	111.00
1.67	3.94	105.8
2.33	2.50	99.45
3.33	2.07	80.13

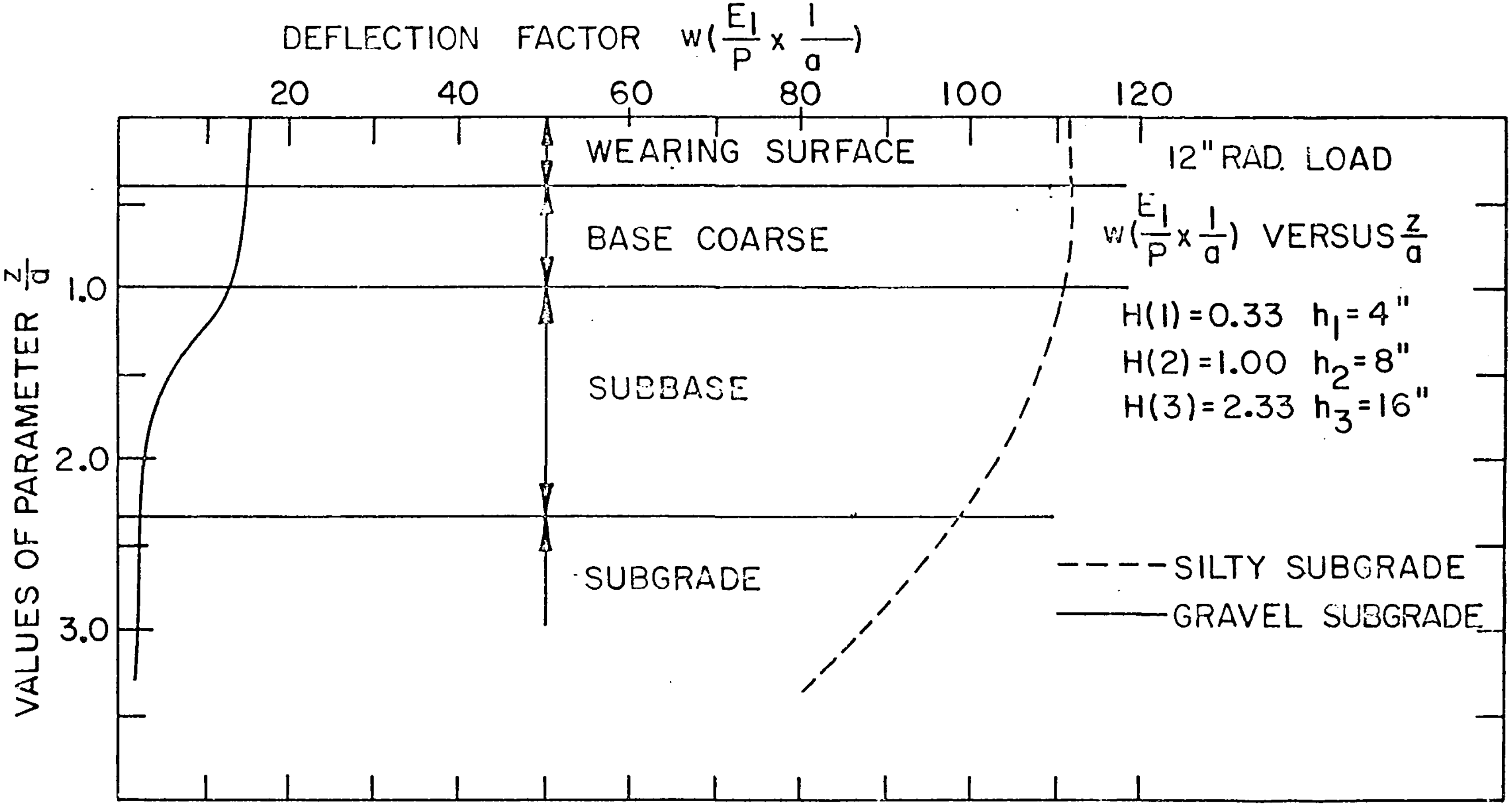

Figure 13. Deflection factor versus z/a

Vertical stress influence coefficient $\sigma_{z/p}$

Tire imprint radius = 12 inches

$\beta_1 = 3.73$	$\beta_2 = 11.43$	$\beta_3 = 11.43$
$H_1 = 0.50$	$H_2 = 1.5$	$H_3 = 3.00$
$h_1 = 6"$	$h_2 = 12"$	$h_3 = 18"$

z/a	Gravel subgrade	Silty subgrade
0.00	1.000	1.000
0.12	0.919	0.918
0.25	0.745	0.742
0.37	0.572	0.570
0.50	0.496	0.438
0.88	0.465	0.413
1.21	0.391	0.245
1.50	0.236	0.128
2.25	0.0946	0.048
3.00	0.0415	0.022
4.00	0.0276	0.016

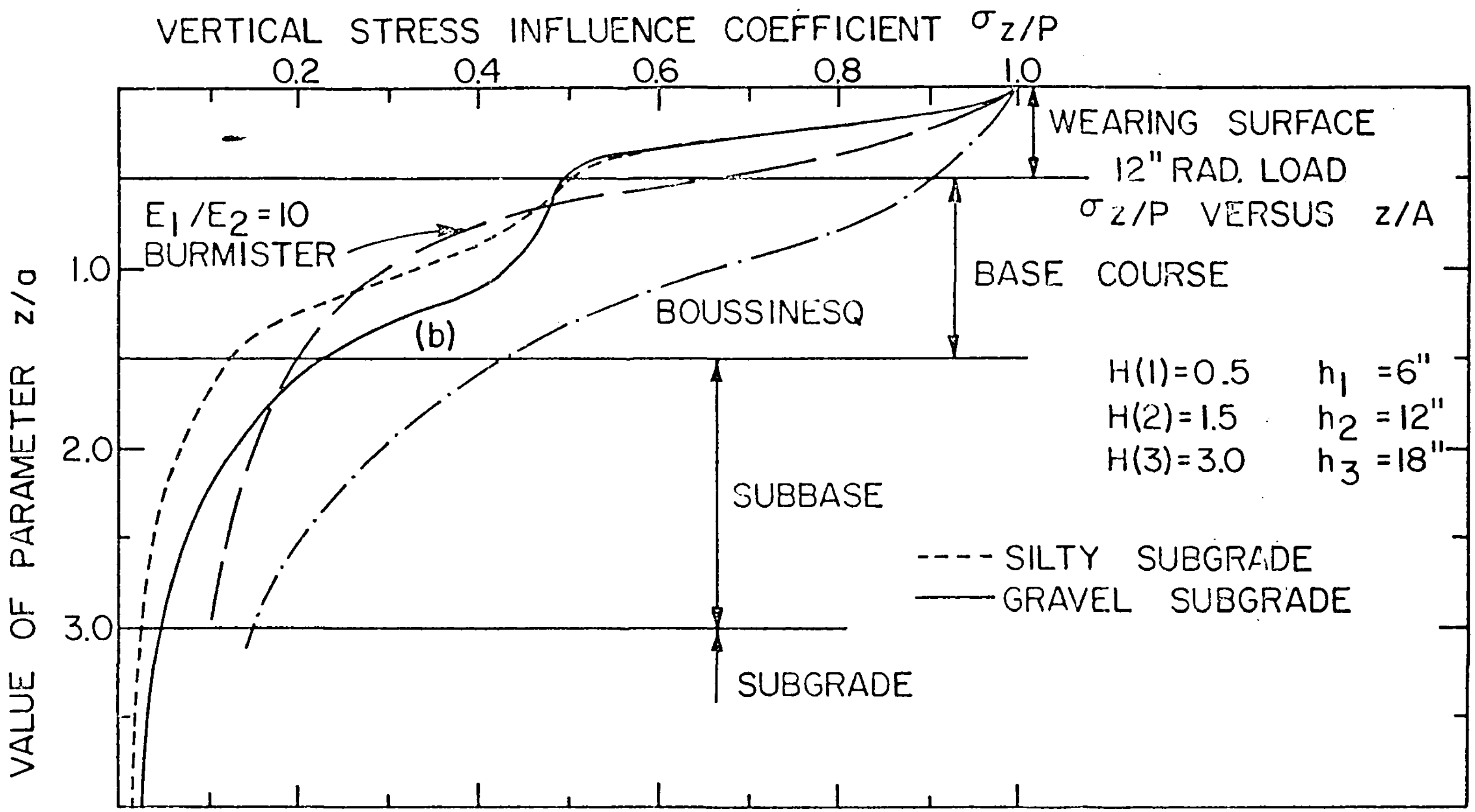

Figure 14. Vertical stress influence coefficient versus $\sigma_{z/a}$

Vertical deflection factor $w(\frac{E_1}{p} \times \frac{1}{a})$

Tire imprint radius = 12 inches

$\beta_1 = 3.73$	$\beta_2 = 11.43$	$\beta_3 = 11.43$
$H_1 = 0.50$	$H_2 = 1.5$	$H_3 = 3.00$
$h_1 = 6"$	$h_2 = 12"$	$h_3 = 18"$

z	Gravel subgrade	Silty subgrade
0.00	11.79	74.43
0.12	11.74	74.37
0.25	11.68	74.36
0.37	11.61	74.20
0.50	11.41	74.26
0.88	10.53	74.14
1.21	9.53	72.66
1.50	8.93	68.60
2.25	2.01	65.00
3.00	1.68	60.00
4.00	1.63	55.00

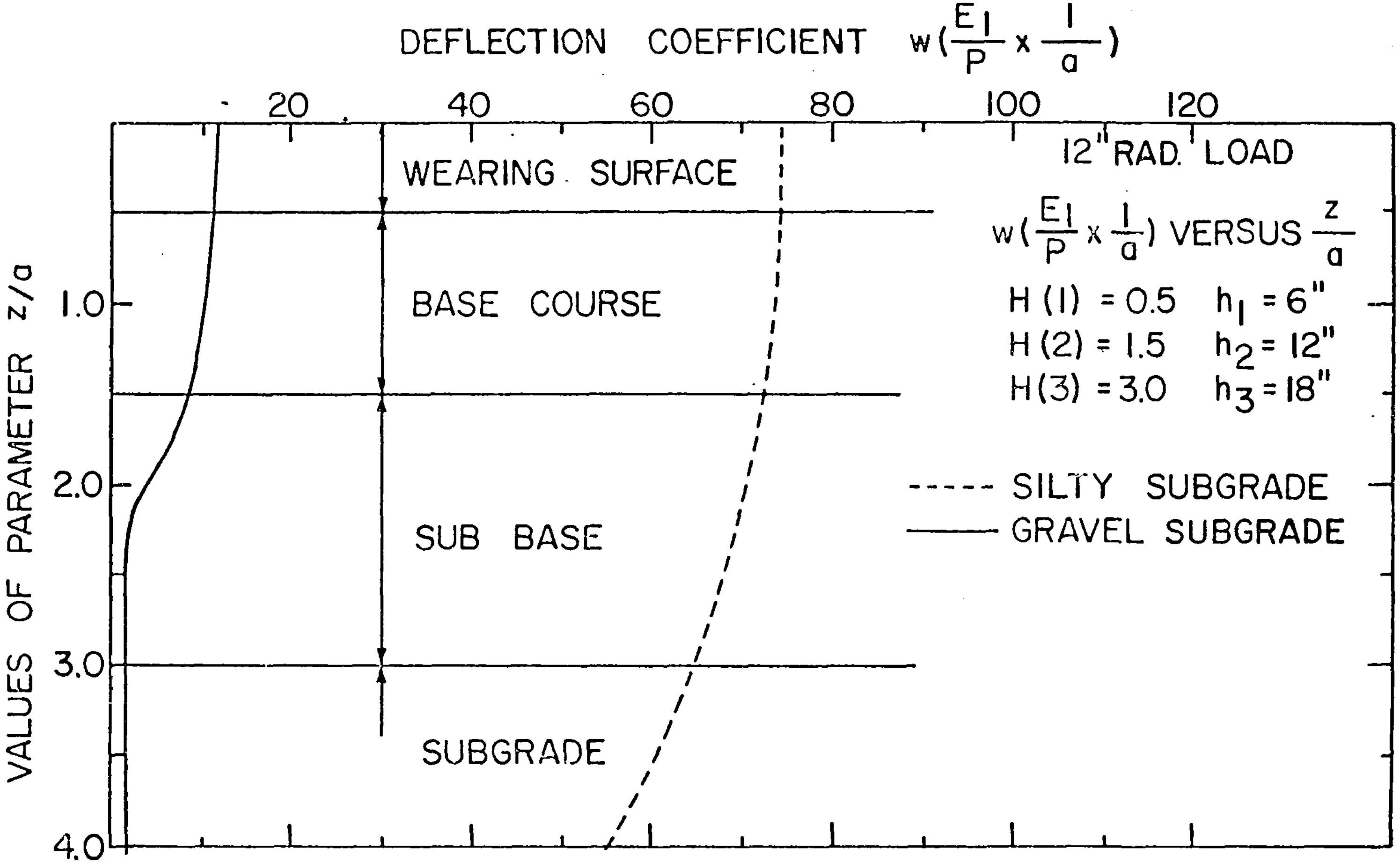

Figure 15. Deflection coefficient versus z/a

An Approach to the Design of Flexible Pavements

The main object of a road structure is to prevent the underlying soil from being subjected to excessive loads produced by traffic. Excessive loads produce excessive deformation of the subgrade which leads to cracking of the surface layers which in turn deteriorates the road structure due to the entrance of water. Sometimes cracks develop at the bottom of a bituminous layer and propagate upwards, due to the horizontal stresses and deformations at the bottom of layers. Therefore the maximum vertical stress in the subgrade and vertical displacement at the surface of the road structure are important design criteria. The vertical stresses and displacements and maximum horizontal stresses and deformations in each layer should be kept within safe limits to prevent cracks due to flexture.

A safe stress for any layer is a function of the strength of the material. As yet there is no fundamental method of evaluating the true permissible stresses for the base materials or for any soil. It is hoped that the stress-strain relationships for these materials which are being calculated from velocity of propagation measurements made in the field by Heukelom and Foster (29) may produce values which will closely approach the true permissible values of the vertical stresses. Presently a relationship between an empirical permissible value and a parameter--the

C.B.R. value which represents a measure of the strength or quality of the soil, can be taken as a satisfactory measure.

Design curves relating the C.B.R. value of the soil with total thickness of the conventional type pavements developed from actual observations on roads are now in use. The graphs for heavy traffic (12,000 wheel load, 60 psi) and for light load (7,000 lbs wheel load, 60 psi) for 2 to 4 inches of bituminous surfacing on a base course of granular material is given in Figure 2 of Peattie's paper (30). These show plots of vertical stresses, existing at the top of the subgrade, versus C.B.R. values for some of the highways in California. Since these represent highways that are performing satisfactorily, the graph represents stresses that can be carried safely by subgrades having these C.B.R. values. Since these stresses can be much lower than the value the pavement may be able to carry, they represent the presumptive values. A modified curve based on further analysis is presented by Porter (32). A similar relationship connecting the vertical deformation with the C.B.R. values of the subgrade can be drawn. These graphs can be used for the design.

Effect of repetitive loads

Road structures fail not only due to excessive stresses under the traffic but also due to fatigue under repeated loads. Pell, McCarthy and Gardner (31) suggest that the principal tensile strain is the critical factor in the fatigue of bituminous mixes. It can be seen from a graph prepared by Saal and Pell and presented by Peattie (28) that there is a linear relationship between the logarithm of the tensile strain and the logarithm of the number of load applications to failure for sand asphalt specimens subjected to uni-directional and bi-directional stresses. Similar work of fatigue behavior for specimens containing larger sized aggregates subjected to three dimensional stresses as they exist in the field is also being done. Bituminous mixes take from 60 to 100 Kg/sg cms. without immediate failure. Since it has bee noticed that the strength decreases to about 75% of its original value after repeated application of loads, it is safe to assume the ultimate tensile strength of an asphaltic mix under field conditions is about 65 Kg/sg cms. From laboratory work on the stress-strain relationships for granular sub-grades it has been found that below certain stresses the number of load application does not bring about fatigue for the material and the vertical stresses in the graph presented by Porter (30), represent these limiting stresses for the corresponding C.B.R. values.

Method of design

It has been concluded that the critical conditions for thickness design are the vertical stress and displacement at the top of the layers and horizontal strain and stress at the bottom of the layer. In a four layer structure the values E_2, E_3 and E_4 are relatively independent of temperature, but E_1 of the bituminous layer depends considerably on time and temperature. The evaluation of stresses and strains should be made at the critical conditions of high temperature and slow traffic. A design approach for a four layer system can be developed on the same lines as has been given by Peattie (28) for a three layer system, which is discussed below in detail.

$$A = \frac{a}{h_2} \qquad H = \frac{h_1}{h_2}$$

$$K_1 = \frac{E_1}{E_2} \qquad K_2 = \frac{E_2}{E_3} \qquad K_3 = \frac{E_3}{E_4}$$

where a = radius of circular contact area

h_1, h_2 and h_3 = thickness of the three layers respectively.

E_1, E_2, E_3 and E_4 = elastic modulii of the four layers from top to bottom.

Ranges of these parameters should cover all the combinations most likely to occur in actual highway pavements. Intermediate values can be interpolated. The numerical data available is not considered sufficient for this purpose but an example is shown by using extrapolated values from the graphs so that, when all combinations of K's, A's and

H's are covered, design curves can be made. The design method suggested is a trial and error process. First assume the layers to be of certain thickness and calculate the stresses and strains under design loads. If these values are not within the permissible limits the thicknesses are adjusted.

A large range of subgrade stresses ZZ_2 for a certain combination of K_1 and K_2 in a three layer system is represented as shown by Peattie (28) in Figure 16. Similar charts can be drawn for surface displacements, horizontal stresses and strains. These charts are the same irrespective of tire pressure. But the charts are different for different values of K_1 and K_2. Now knowing the values of C.B.R. for the subgrade and other layers and referring to graphs similar to the one due to Saal and Pell, we can find permissible stresses and strains. If the ratio of the permissible stress to tire pressure is M then all pavements represented by A and H below the stress factor M in Figure 16 are safe. Similar permissible values should be entered in the corresponding charts of surface displacements of Figure 17. The horizontal stresses and strains should also be checked on similar graphs prepared. The common values of A and H marked by x in both figures represent a structure where the vertical stress and horizontal strain reach the permissible values simultaneously. The structures represented

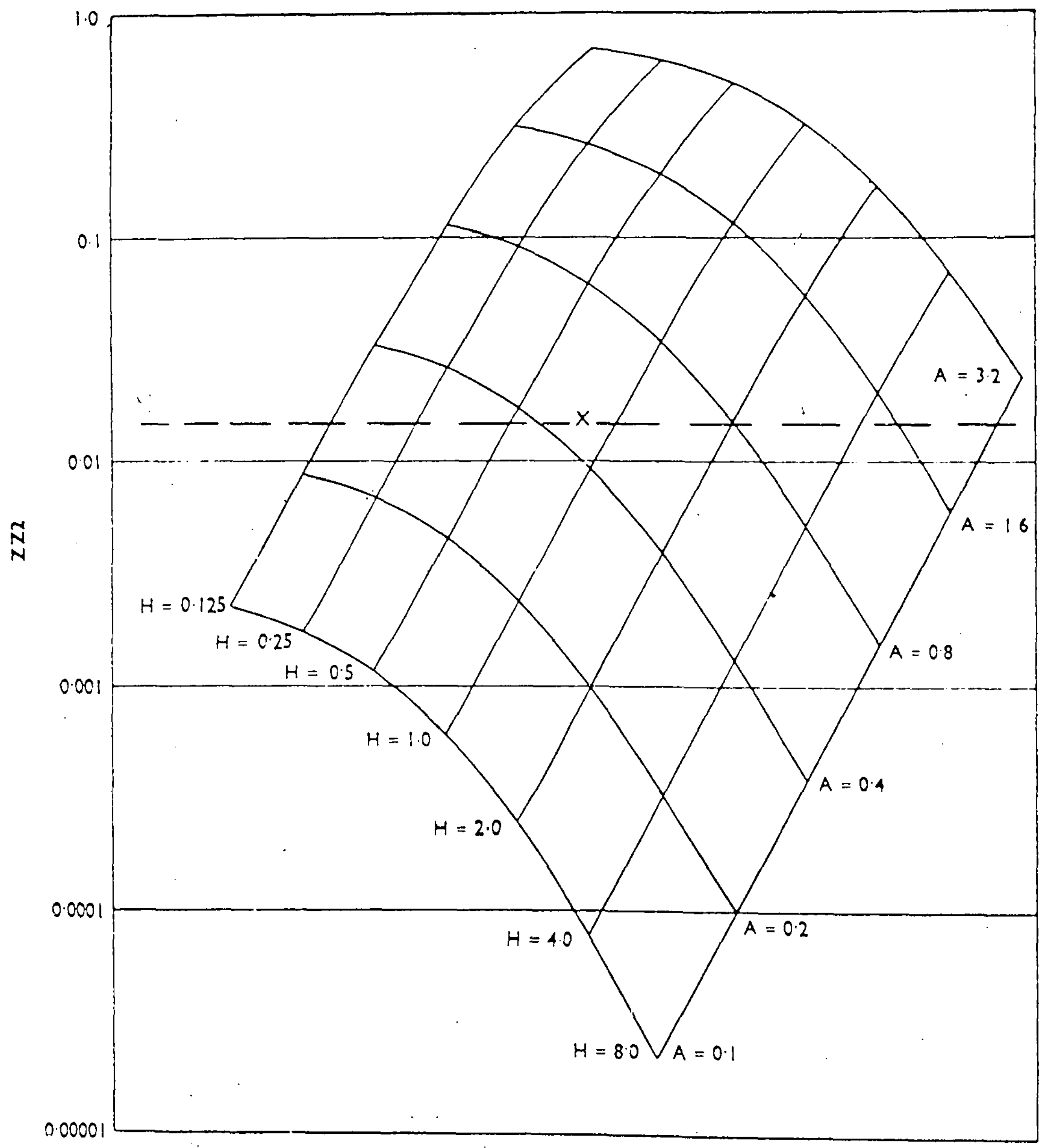

Figure 16. Factor for vertical compressive stress factor zz_2 for $k_1=2$ and $k_2=20$

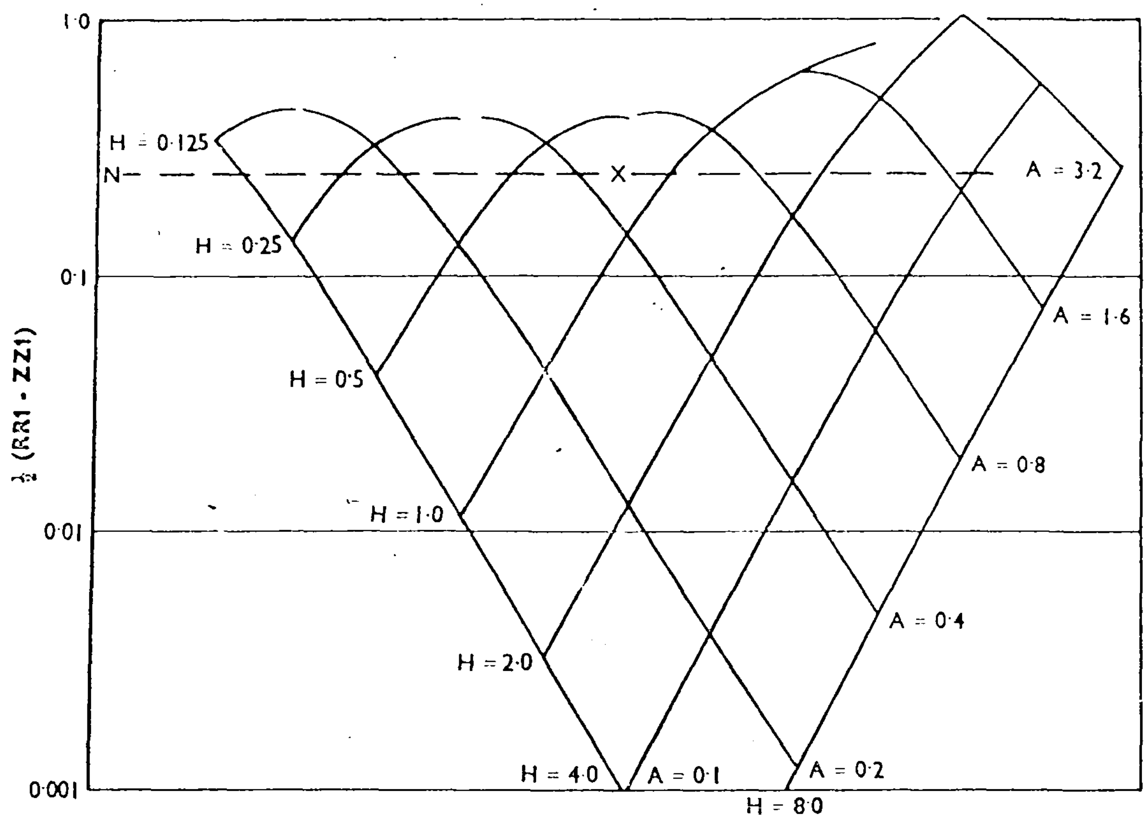

Figure 17. Factor for horizontal strain factor $\frac{1}{2}$ $(RR_1 - ZZ_1)$ for $k_1=2$ and $k_2=20$

by A and H on the right side of point x in Figure 16 reach a permissible stress before they reach the permissible horizontal strain. The opposite is true of points on the left side of point x. The satisfactory values are those represented by points that are common to the areas under the horizontal lines through x in both Figures 16 and 17. For a four layer system there are many variables and the process is tedious but practicable. A chart similar to Figure 16 is shown as an example in Figure 18. For each set of K_1, K_2 and K_3 since there are 3 factors f, g, h where

$$f = \frac{h_1}{a} \qquad g = \frac{h_2}{a} \qquad h = \frac{h_3}{a}$$

the two dimensional nature of charts is a disadvantage but by using a trial and error procedure a small number of charts should be sufficient for the solution of a wide latitude of problems.

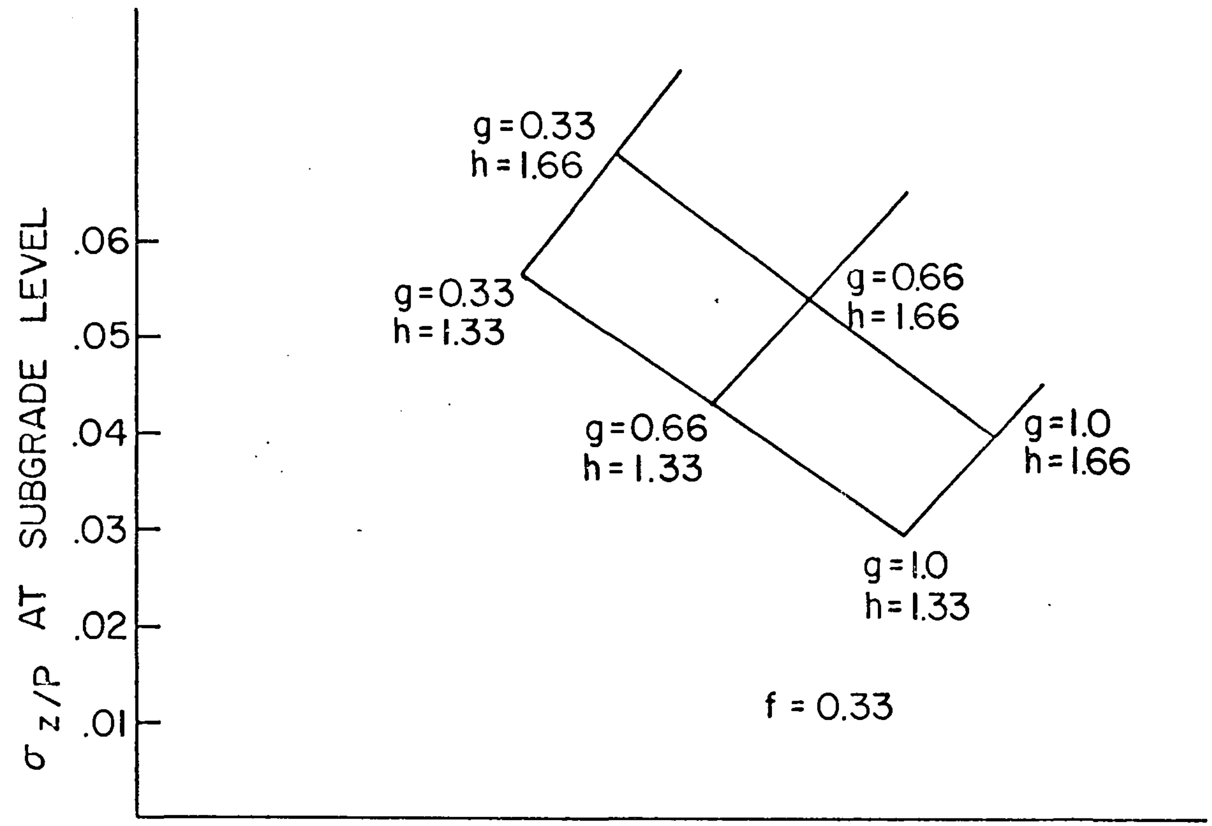

Figure 18. Vertical stress influence coefficient $\sigma_{z/p}$ at subgrade level versus g, and h for f=0.33

SUMMARY

The following general conclusions can be drawn by comparing the vertical displacements and normal stresses of different pavements.

The displacement of the pavement on silt is smaller than that of the pavement on gravel at all depths. This may be due to the smaller elastic modulus of silt. The load-deflection factor improves markedly with the increase in thicknesses of the pavement layers, causing a decrease in the displacement. Improvement in deflection performance can be achieved by using high quality materials with higher moduli of elasticity and by actual constructional excellence in the field to attain full potential strength properties. The deflections are also influenced by the radius of the bearing area. For a constant intensity of pressure the deflection increases as the tire imprint increases in area.

While the surface displacement is considerably affected by the elastic modulus of the subgrade, the graphs show that the stress influence coefficients in the subgrade do not change significantly with a change in the elastic modulus of the subgrade. The stresses mainly depend on the thicknesses of the pavement layers and the elastic moduli of the pavement layers since a major part of the load is carried by the pavement. For comparison the stress influence coefficient curves of Boussinesq and Burmister have been plotted on Figure 14. It can be seen that the stress influence coefficients for all cases are lower than that of Boussinesq at all depthes, and that for this case they are nearly the same as that for a two layer system with $E_1/E_2 = 10$ and $H_1 = 1 \cdot 0$.

The stresses and displacements are affected by many factors such as the absolute values and ratios of the elastic modulii of consecutive layers, the radius of bearing area and thicknesses of the different layers of the pavement. If road test results are to be extrapolated, it seems logical that theoretical rather than empirical relationships would provide a better foundation for such extrapolation. The best design results will be obtained by evaluating the stresses and displacements for certain combinations of the above-mentioned factors and extrapolating only within a reasonable range.

Considerable experimental work needs to be done in this field before a reliable pavement design procedure can be developed from these theoretical values. Precise values of the "β" and the elastic modulii of the layers need to be evaluated, horizontal and vertical stress values for many different pavement thicknesses should be calculated, and model studies to check predicted stress values will be invaluable.

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NOMENCLATURE

A_j, B_j, C_j, D_j = Constants of integration for the j^{th} layer.

E_j = Modulus of elasticity of the material in the j^{th} layer.

$F(\xi)$ = Hankel transform of the distribution of tire pressure.

H = Distance from the upper surface of the layered system to an interface divided by the radius of the area of load distribution on the surface.

h_j = Thickness of the j^{th} layer in inches.

J_o, J_1 = Bessel functions of the first kind, order zero and one respectively.

j = Subscript referring to quantities corresponding to the j^{th} layer.

n = Number of layers in a given system.

Q = Coefficient matrix.

(r, θ, z) = Cylindrical co-ordinates

S = Coefficient matrix

s = Number of points in the Gaussian formula.

(u, o, w) = Displacements of a point in the r, θ, z directions.

u_j, $\overline{u}_j$ = Radial displacement of the points in the j^{th} layer and its Hankel transform respectively.

W(i) = Weighting coefficients in the Gaussian formula.

w_j, $\overline{w}_j$ = Vertical displacement of the points in the j^{th} layer and its Hankel transform.

u_x, v_y, w_z = Displacements in the x, y and z directions.

Y_o = Column matrix (A_j, B_j, C_j, D_j).

(x, y, z) = Rectangular co-ordinates.

β = Proportionality constant, between shear stress and relative displacement at the interface.

ϵ_x, ϵ_y, ϵ_z = Strain components in the x, y, and z directions.

θ = Angle between r and x directions.

(ϵ_r, ϵ_θ, ϵ_z) = Strain components in the radial, circumferential and vertical directions.

γ_{xy}, γ_{yz}, γ_{zx} = Shear strains in the rectangular co-ordinates.

γ_{rz}, $\gamma_{r\theta}$, $\gamma_{\theta z}$ = Shear strains in the cylindrical co-ordinates.

$\Delta = \epsilon_r + \epsilon_\theta + \epsilon_z$

$\Phi = \sigma_r + \sigma_\theta + \sigma_z = \sigma_n + \sigma_y + \sigma_z$

λ, μ = Lame's constants.

ν = Poisson's ratio.

ξ = Variable of integration introduced by Hankel transformation in the equations of stresses and strains.

σ_x, σ_y, σ_z = Normal stresses in the x, y, z directions due to external loads.

σ'_z, σ'_r, σ'_θ = Vertical, radial and circumferential stresses due to the weight of the material.

σ_z, σ_r, σ_θ = Vertical radial and circumferential stresses due to external loads.

$(\sigma_r)_j$, $(\sigma_\theta)_j$, $(\sigma_z)_j$ = Radial, circumferential and vertical components of stress in the j^{th} layer.

$(\bar{\sigma}_r)_j$, $(\bar{\sigma}_\theta)_j$, $(\bar{\sigma}_z)_j$ = Hankel transforms of the radial, circumferential and vertical components of stress of points in the j^{th} layer.

σ_x, σ_y, σ_z = Stress components in the x, y, z directions.

$(\tau_{rz})_j$, $(\overline{\tau}_{rz})_j$ = Shear stress for the points in the j^{th} layer and its transform respectively.

$(\tau_{r\theta}$, $\tau_{\theta z}$, τ_{rz} = The shear stress on planes perpendicular to z, r and θ.

$(\tau_{ry}$, τ_{yz}, $\tau_{xz})$ = The shear stress on planes perpendicular to z, x, y directions

∇^2 = Laplace operator $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$